

# The Gap of the Consecutive Eigenvalues of the Drifting Laplacian on Metric Measure Spaces

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## Abstract

In this paper, we investigate eigenvalues of the Dirichlet problem and the closed eigenvalue problem of drifting Laplacian on the complete metric measure spaces and establish the corresponding general formulas. By using those general formulas, we give some upper bounds of consecutive gap of the eigenvalues of the eigenvalue problems, which is sharp in the sense of the order of the eigenvalues. As some interesting applications, we study the eigenvalue of drifting Laplacian on Ricci solitons, self-shrinkers and product Riemannian manifolds. We give the explicit upper bounds of the gap of the consecutive eigenvalues of the drifting Laplacian. Since eigenvalues is invariant in the sense of isometry, by the classifications of Ricci solitons and self-shrinkers, we give the explicit upper bounds for the consecutive eigenvalues of the drifting Laplacian on a large class metric measure spaces. In addition, we also consider the case of product Riemannian manifolds with certain curvature conditions and some upper bounds are obtained. Basing on the case of Laplace operator, we also present a conjecture as follows: all of the eigenvalues of the Dirichlet problem of drifting Laplacian on metric measure spaces satisfy:

$$\lambda_{k+1} - \lambda_k \leq (\lambda_2 - \lambda_1)k^{\frac{1}{n}}.$$

We note the conjecture is true in some special cases.

**Keywords:** drifting Laplacian; metric measure space; consecutive eigenvalues.

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## 1 Introduction

Let  $M^n$  be an  $n$ -dimensional complete Riemannian manifold with smooth metric  $g$ , and  $\Omega$  is a bounded domain with piecewise smooth boundary  $\partial\Omega$ . We consider the following Dirichlet problem:

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where

$$\Delta = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1} \partial_i g^{ji} \sqrt{\det(g)} \partial_j.$$

If  $M^n$  is an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , L. E. Payne, G. Pólya and H. F. Weinberger [64] and [66] investigated the eigenvalue inequalities of the Dirichlet problem (1.1). They established the following universal inequality:

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^k \lambda_i. \quad (1.2)$$

In various backgrounds, many mathematicians extended Payne, Pólya and Weinberger's universal inequality. However, among a large amount of literatures, there are two main contributions due to G. N. Hile and M. H. Protter [40] and H.-C. Yang [82]. In 1980, G. N. Hile and M. H. Protter proved the following universal inequality of eigenvalues:

$$\sum_{i=1}^k \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}. \quad (1.3)$$

After a direct calculation, one can show that inequality (1.3) implies inequality (1.2). In 1991, H.-C. Yang proved a very sharp universal inequality in his famous paper [82] (cf. [19]):

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i, \quad (1.4)$$

which is called H.-C. Yang's first inequality by M. S. Ashbaugh (cf. [2], [3]). From (1.4), one can infer that

$$\lambda_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^k \lambda_i, \quad (1.5)$$

which is called H.-C. Yang's second inequality (cf. [2], [3]). In 2007, Q.-M. Cheng and H.-C. Yang established a celebrated recursion formula [19]. By utilizing this recursion formula, they gave an explicit upper bound:

$$\lambda_{k+1} \leq C_0(n, k) k^{\frac{2}{n}} \lambda_1, \quad (1.6)$$

where the constant  $C_0(n, k) \leq 1 + \frac{4}{n}$  only depend on  $n$  and  $k$  (see Q.-M. Cheng and H.-C. Yang's paper [19]). Let  $\Omega$  be a bounded domain on an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  or hyperbolic space. For this assumption, in 2016, D. Chen, T. Zheng and H.-C. Yang [14] obtained an upper for the gap of consecutive eigenvalues of the eigenvalue problem (1.1) as follows:

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}}, \quad (1.7)$$

where

$$C_{n,\Omega} = 4\lambda_1 \sqrt{\frac{C_0(n)}{n}},$$

and the constant  $C_0(n)$  is the same as the one in (1.6). It is well known that the order of the upper bound of the gap of the consecutive eigenvalues of  $\mathbb{S}^n$  with standard metric is  $k^{\frac{1}{n}}$ . Therefore, for general Riemannian manifolds, D. Chen, T. Zheng and H.-C. Yang proposed the following conjecture in the same paper [14]:

**Conjecture 1.1.** *Let  $(M^n, g, f)$  be a complete smooth measure space and  $\lambda_i$  be the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the eigenvalue problem (1.1). Then we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}},$$

where

$$C_{n,\Omega} = 4(\lambda_1 + c_1) \sqrt{\frac{C_0(n)}{n}}.$$

Furthermore, by constructing a new trial function, the author recently made an affirmative answer to this conjecture in [86].

Let  $M^n$  be an  $n$ -dimensional closed Riemannian manifold. We consider the closed eigenvalue problem of Laplacian:

$$\Delta u = -\bar{\lambda}u, \quad \text{in } M^n. \quad (1.8)$$

It is well known that the eigenvalues of the closed eigenvalue problem (1.8) is discrete and satisfies the following:

$$0 = \bar{\lambda}_0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3 \leq \dots \leq \bar{\lambda}_k \leq \dots \rightarrow +\infty,$$

where  $\bar{\lambda}_k$  is the  $k$ -th eigenvalue of the closed eigenvalue problem (1.8) and each eigenvalue is repeated according to its multiplicity. We assume that  $M^n$  is an  $n$ -dimensional compact homogeneous Riemannian manifold. In 1980, P. Li [47] investigated the closed eigenvalue problem (1.8) and proved the following universal inequality:

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq \frac{2}{k+1} \left( \sqrt{\left( \sum_{i=1}^k \bar{\lambda}_i \right)^2 + (k+1) \sum_{i=1}^k \bar{\lambda}_i \bar{\lambda}_1 + \sum_{i=1}^k \bar{\lambda}_i} \right) + \bar{\lambda}_1.$$

If  $M^n$  is an  $n$ -dimensional compact minimal submanifold in a unit sphere  $\mathbb{S}^N(1)$ , then, in 1980, P. C. Yang and S. T. Yau [83] proved the eigenvalues of the closed eigenvalue problem satisfy the following eigenvalue inequality:

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq n + \frac{2}{n(k+1)} \left( \sqrt{\left( \sum_{i=1}^k \bar{\lambda}_i \right)^2 + n^2(k+1) \sum_{i=1}^k \bar{\lambda}_i \bar{\lambda}_1 + \sum_{i=1}^k \bar{\lambda}_i} \right).$$

Furthermore, E. M. Harrel II and P. L. Michel and J. Stubbe (see ([35] 1994) and ([36] 1997)) obtained an abstract inequality of algebraic version. By applying the algebraic inequality, they proved

that, if  $M^n$  is an  $n$ -dimensional compact minimal submanifold in a unit sphere  $\mathbb{S}^N(1)$ , one has the following eigenvalue inequality:

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq n + \frac{4}{n(k+1)} \sum_{i=1}^k \bar{\lambda}_i, \quad (1.9)$$

and if  $M^n$  is an  $n$ -dimensional compact homogeneous Riemannian manifold, then we have

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq \frac{4}{k+1} \sum_{i=1}^k \bar{\lambda}_i + \bar{\lambda}_1, \quad (1.10)$$

One can easily see that the above inequalities (1.9) and (1.10) made significant improvement to earlier estimates of differences of consecutive eigenvalues of Laplacian introduced by P. C. Yang and S. T. Yau [83], P.-F. Leung [46], P. Li [47] and E. M. Harrel II [34]. Q.-M. Cheng and H.-C. Yang also considered the same eigenvalue problem and proved that, when  $M^n$  is an  $n$ -dimensional compact homogeneous Riemannian manifold without boundary, then the eigenvalues of the close eigenvalue problem (1.8) satisfy

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq \left[ \left( \frac{4}{k+1} \sum_{i=1}^k \bar{\lambda}_i + \bar{\lambda}_1 \right)^2 - \frac{20}{k+1} \sum_{i=0}^k \left( \bar{\lambda}_i - \frac{1}{k+1} \sum_{j=1}^k \bar{\lambda}_j \right)^2 \right]^{\frac{1}{2}};$$

and when  $M^n$  is an  $n$ -dimensional compact minimal submanifold without boundary in a unit sphere  $\mathbb{S}^N(1)$ , then the eigenvalues of the close eigenvalue problem (1.8) satisfy

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq 2 \left[ \left( \frac{2}{n} \frac{1}{k} \sum_{i=0}^k \bar{\lambda}_i + \frac{n}{2} \right)^2 - \left( 1 + \frac{4}{n} \right) \frac{1}{k+1} \sum_{j=0}^k \left( \bar{\lambda}_j - \frac{1}{k} \sum_{i=0}^k \bar{\lambda}_i \right)^2 \right]^{\frac{1}{2}}.$$

In [86], the author studied the closed eigenvalue problem (1.8) of Laplacian and obtained a similar optimal upper bound. As a further interest, the author also investigated the eigenvalues of the Laplacian on compact homogeneous Riemannian manifolds without boundary in [86].

We suppose that  $f$  is a smooth function on  $M^n$ . The triple  $(M^n, g, e^{-f} dv)$  is called a metric measure space with weighted volume density  $e^{-f} dv$ . Furthermore, we say that the triple  $(M^n, g, e^{-f} dv)$  is an  $n$ -dimensional complete metric measure space if  $M^n$  is a complete Riemannian manifold with dimension  $n$ , while the triple  $(M^n, g, e^{-f} dv)$  is an  $n$ -dimensional closed metric measure space if  $M^n$  is a closed Riemannian manifold with dimension  $n$ . The metric measure spaces also arise in smooth collapsed Gromov-Hausdorff limits. So-called Bakry-Émery Ricci tensor  $\text{Ric}^f$  corresponding to weighted metric measure spaces is a very important curvature quantity, which is defined by

$$\text{Ric}^f := \text{Ric} + \text{Hess} f, \quad (1.11)$$

where  $\text{Ric}$  and  $\text{Hess} f$  denote Ricci tensor of  $M^n$  and Hessian of  $f$ , respectively (see [4, 49]). When  $f$  is a constant, we have

$$\text{Ric}^f = \text{Ric}. \quad (1.12)$$

Therefore, the Bakry-Émery Ricci tensor is naturally viewed as an extension of the Ricci tensor. Recently, a great deal of significant results under assumption on the Bakry-Émery Ricci tensor have been obtained. For instances, A. Lichnerowicz [50, 51] has extended the classical Cheeger-Gromoll splitting theorem to the metric measure spaces with  $\text{Ric}^f \geq 0$  and  $f$  is bounded, G. F. Wei and W. Wylie in [78] have proved the weighted volume comparison theorems; O. Munteanu and J. Wang [60, 61] have established gradient estimates for positive weighted harmonic functions. The metric measure space has studied by many geometric analysis (cf: [1, 6–8, 16, 60, 61, 75, 80]) during the last twenty years. Next, we give definition of the drifting Laplacian associated with the metric measure space:

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle = e^f \operatorname{div} (e^{-f} \nabla u).$$

It is not difficult to see that drifting Laplacian is a self-adjoint operator with respect to the weighted volume measure  $e^{-f} dv$ , i.e.,

$$- \int_{M^n} \langle \nabla u, \nabla w \rangle e^{-f} dv = \int_{M^n} u (\Delta_f w) e^{-f} dv = \int_{M^n} w (\Delta_f u) e^{-f} dv, \quad (1.13)$$

and it is an important elliptic operator which is widely used in the probability theory and geometrical analysis. In particular, many mathematicians pay more and more attention to the research of eigenvalue of the drifting Laplacian in recent years. For this recent developments, we refer to [1, 11, 29, 49, 54, 60, 61, 74, 78, 79] and the references therein. On one hand, L. Ma and S.-H. Du [54] and H. Li and Y. Wei [49] have studied the Reilly formula of the Witten-Laplacian version to obtain a lower bound of the first eigenvalue for the Witten-Laplacian on the  $f$ -minimal hypersurface. Furthermore, they have given a Lichnerowicz type lower bound for the first eigenvalue of the Witten-Laplacian on compact manifolds with positive Bakry-Émery Ricci curvature. In 2013, A. Futaki and Y. Sano [28] have studied the lower bound of the first eigenvalue of the Witten-Laplacian on compact manifolds  $M^n$  if the Bakry-Émery Ricci curvature bounded from below by  $(n-1)K$  and obtained the following:

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31K}{100}, \quad (1.14)$$

and A. Futaki, H. Li and X.-D. Li [29] (cf. [1]) have also improved the above result to

$$\lambda_1 \geq \sup_{s \in (0,1)} \left\{ 4s(1-s) \frac{\pi^2}{d^2} + sK \right\},$$

where  $d$  is the diameter of  $(M^n, g)$ . As an application, an upper bound of the diameter of  $(M^n, g)$  has been obtained. In addition, under the assumption  $\text{Ric}_f \geq -(n-1)k$  for some  $k \geq 0$ , N. Charalambous, Z. Lu and J. Rowlett obtained [10]:

$$\lambda_1 \geq \frac{\pi^2}{d^2} \exp(-c_n \sqrt{kd^2}), \quad (1.15)$$

where  $d$  is the diameter of  $M$  with respect to  $g$ , and  $c_n$  is a constant depending only on  $n$ . In [10], N. Charalambous, Z. Lu and J. Rowlett proved the Bakry-Émery maximum principle. Applying this

result, they proved the eigenvalue inequality (1.14) given by A. Futaki and Y. Sano [27]. We note that the corresponding Riemannian case is proved by J. Ling [52]. On the other hand, upper bounds for the first eigenvalue of the drifting Laplacian on complete Riemannian manifolds have been studied in [60, 61, 74, 79]. In particular, J. Wu in [79] (also see [80]) established an upper bounds for the first eigenvalue of the drifting Laplacian on compact gradient Ricci soliton if  $f$  is bounded. Assume that  $(M^n, g, f)$  is a compact metric measure space without boundary, and  $\epsilon > 0$ . If

$$Ric_f - \epsilon \nabla f \otimes \nabla f \geq -(n-1)K, \text{ for } K \geq 0,$$

then we have (see [10]) the following estimate:

$$\lambda_k \leq C(n, \epsilon)(K + k^2/d^2), \forall k \in \mathbb{N},$$

where  $d$  is the diameter of  $M$  and  $C(n, \epsilon)$  is a constant depending on  $n$  and  $\epsilon$ . Furthermore, we assume that  $K = 0$ , then, by using make of (1.15), we have

$$\lambda_k \leq C(n, \epsilon)\lambda_1. \quad (1.16)$$

If  $(M, g, f)$  is a compact Bakry-Émery manifold with non-negative Bakry-Émery Ricci curvature, then, in 2013, K. Funano and T. Shioya proved [27] the following stronger and somewhat surprising inequality:

$$\lambda_k \leq C_k \lambda_1,$$

where  $C_k$  is a positive constant which depends only on  $k$  and in particular is independent of  $(M, g, f)$ . Using an example, K. Funano and T. Shioya showed that the non-negativity of curvature is a necessary condition (see [27]). The proof relies on a geometric theory of concentration of metric measure spaces due to M. Gromov [30]. We also note that A. Hassannezhad demonstrated upper bounds for the eigenvalues without curvature assumptions [38].

In this paper, we consider the following Dirichlet problem of drifting Laplacian:

$$\begin{cases} \Delta_f u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.17)$$

where  $\Omega \subset M^n$  is a bounded domain with piecewise smooth boundary  $\partial\Omega$  in an  $n$ -dimensional complete metric measure space  $(M^n, g, e^{-f})$ . It is clear that eigenvalue problem (1.17) is exactly eigenvalue (1.1) when  $f$  is a constant. If  $\lambda_i$  is the  $i$ -th eigenvalue of this problem, then the spectrum of the Dirichlet problem (1.17) is discrete and satisfies the following:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty,$$

where each eigenvalue is repeated according to its multiplicity. All through this paper, we always assume that the dimensional  $n$  is larger than one. For this eigenvalue problem, our first result is the following:

**Theorem 1.2.** *Let  $(M^n, g, f)$  be a complete metric measure space, where  $M^n$  is an  $n$ -dimensional complete Riemannian manifold isometrically immersed in a Euclidean space  $\mathbb{R}^{n+p}$ , and  $\lambda_i$  be the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the Dirichlet problem (1.17). Then we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (1.18)$$

where  $C_{n,\Omega,f}$  is a constant dependent on  $\Omega$  itself and the dimension  $n$ .

In this paper, we also investigate the eigenvalues of the closed eigenvalue problem of drifting Laplacian on compact Riemannian manifolds:

$$\Delta_f u = -\bar{\lambda} u. \quad (1.19)$$

Spectrum of the closed eigenvalue problem (1.19) is discrete and satisfies

$$0 = \bar{\lambda}_0 \leq \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \dots \leq \bar{\lambda}_k \leq \dots \rightarrow +\infty,$$

where each eigenvalue is repeated according to its multiplicity.

Similarly, we assume that  $(M^n, g, f)$  is an  $n$ -dimensional closed metric measure space, which is isometrically immersed in an  $(n + p)$ -dimensional Euclidean space  $\mathbb{R}^{n+p}$ , then we have the following:

**Theorem 1.3.** *Let  $(M^n, g, f)$  be a closed metric measure space and  $M^n$  an  $n$ -dimensional closed Riemannian manifold isometrically immersed into the Euclidean space  $\mathbb{R}^{n+p}$ . Assume that  $\bar{\lambda}_i$  is the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the closed eigenvalue problem (1.19). Then we have*

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq C_{n,M^n,f} k^{\frac{1}{n}}, \quad (1.20)$$

where  $C_{n,M^n,f}$  is a constant dependent on  $M^n$  itself, function  $f$ , and the dimension  $n$ .

*Remark 1.1.* In theorem 1.2 and theorem 1.3, the constants  $C_{n,\Omega,f}$  and  $C_{n,M^n,f}$  are allowed to be different in different backgrounds.

In 1982, R. S. Hamilton introduced Ricci solitons [31, 32], which are self-similar solutions to the Ricci flow. Because Ricci solitons represent the fixed points of the Ricci flow, they are an important object in understanding the Ricci flow. Ricci solitons is an important example of complete metric measure space, which is defined as follows: Let  $M^n$  be a complete Riemannian manifold with smooth metric  $g = (g_{ij})$ , then  $(M^n, g, f)$  is called a gradient Ricci soliton if there is a constant  $\rho$  such that

$$R_{ij} + f_{ij} = \rho g_{ij}, \quad (1.21)$$

where  $R_{ij}$  and  $f_{ij}$  denote components of the Ricci tensor and Hessian of  $f$ , respectively. The Ricci soliton is said to be shrinking, steady and expanding according as  $\rho > 0$ ,  $\rho = 0$  or  $\rho < 0$ , respectively. The function  $f$  is called a potential function of the gradient Ricci soliton (cf. [21]). From the equation (1.21), it is not difficult to see that Ricci solitons are generalizations of Einstein metrics. We investigate the eigenvalue of the Dirichlet problem (1.17) of drifting Laplacian on complete noncompact Ricci solitons and prove the following:

**Theorem 1.4.** *Let  $(M^n, g_{ij}, f)$  be an  $n$ -dimensional compact gradient Ricci Soliton. Then, for any  $j$ , eigenvalues of the closed eigenvalue problem (1.8) of drifting Laplacian satisfy*

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq C_{n,M^n,f}(k+1)^{\frac{1}{n}}, \quad (1.22)$$

where

$$C_{n,M^n,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}},$$

$C_0(n)$  is the same as the one in (1.6),

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} \left( n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S \right),$$

and

$$\bar{c} = \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}.$$

Let  $X : M^n \rightarrow \mathbb{R}^{n+p}$  be an  $n$ -dimensional submanifold in the Euclidean space  $\mathbb{R}^{n+p}$ . If  $X : M^n \rightarrow \mathbb{R}^{n+p}$  satisfies

$$n\vec{H} = -X^N,$$

where  $\vec{H}$  and  $X^N$  denote the mean curvature vector and the orthogonal, then we say that it is called a self-shrinker projection of  $X$  into the normal bundle of  $M^n$ , respectively. As another application of the general formula (2.21), we consider the self-shrinker of the mean curvature flow, which is introduced by G. Huisken in [41](cf. T. H. Colding and W. P. Minicozzi [23]).

**Theorem 1.5.** *Let  $H$  and  $X$  denote the mean curvature of  $M^n$  and the position vector of  $M^n$ , respectively. Then, for an  $n$ -dimensional complete self-shrinker  $M^n$  in the Euclidean space  $\mathbb{R}^{n+p}$ , eigenvalues of the Dirichlet problem (1.17) of drifting Laplacian with  $f = \frac{|X|^2}{2}$  satisfy*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,X} k^{\frac{1}{n}},$$

where

$$C_{n,\Omega,X} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}},$$

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} \left( n^2 H^2 + |2n - |X|| + |X|^2 \right),$$

and  $\Psi$  denotes the set of all isometric immersions from  $M^n$  into a Euclidean space.



This paper is organized as follows. In section 2, we prove several key lemmas. By utilizing those key lemmas, we prove a general formula of the eigenvalues of the Dirichlet problem. By the same method, we establish the corresponding general formulas with respect to the closed eigenvalue problem. By utilizing those general formulas, we give the proofs of theorem 1.2 and theorem 1.3 in section 3. In last part of section 3, we give a gap conjectures of consecutive eigenvalues of the Dirichlet problem (1.17) of drifting Laplacian on complete Riemannian manifolds. In section 4, we investigate the eigenvalue of the drifting Laplacian on the complete Ricci solitons. As some further applications, we give the explicit upper bounds for the consecutive eigenvalues of Laplacian on some important Ricci solitons in section 5. As a further interest, we give the explicit upper bounds for the consecutive eigenvalues of Laplacian on self-shrinkers in section 6. In section 7, we consider the eigenvalue problem of drifting Laplacian on splitting Riemannian manifolds. The last section is an appendix, we give the proof of theorem 5.2 in this appendix.

## 2 General formulas for eigenvalues

In this section, we would like to establish some general formulas for eigenvalues, which generalizes a formula of D. Chen, T. Zheng and H.-C. Yang in [14] for the case of Laplacian. Firstly, we shall use the same notations as in [14]. We define  $\mathcal{H}^\infty$  by

$$\mathcal{H}^\infty = \left\{ x = (x_j)_{j=1}^\infty \mid x_j \in \mathbb{R} \left( \sum_{j=1}^\infty x_j^2 \right)^{\frac{1}{2}} < +\infty \right\},$$

with inner product  $\langle \cdot, \cdot \rangle_\infty$ , where  $\langle \cdot, \cdot \rangle_\infty$  is defined by

$$\langle x, y \rangle_\infty = \sum_{j=1}^\infty x_j y_j, \quad \forall x = (x_j)_{j=1}^\infty, y = (y_j)_{j=1}^\infty.$$

Similarly, we can also define  $\mathcal{H}^2$  by

$$\mathcal{H}^2 = \left\{ x = (x_1, x_2) \mid x_1, x_2 \in \mathbb{R} \left( x_1^2 + x_2^2 \right)^{\frac{1}{2}} < +\infty \right\},$$

with inner product  $\langle \cdot, \cdot \rangle_2$ , where  $\langle \cdot, \cdot \rangle_2$  is defined by

$$\langle x, y \rangle_2 = \sum_{j=1}^2 x_j y_j, \quad \forall x = (x_j)_{j=1}^2, y = (y_j)_{j=1}^2.$$

It is not difficult to see that both  $(\mathcal{H}^\infty, \langle \cdot, \cdot \rangle_\infty)$  and  $(\mathcal{H}^2, \langle \cdot, \cdot \rangle_2)$  are Hilbert space. The dual space of  $\mathcal{H}^2$  is denoted by  $(\mathcal{H}^2)^*$ . It is well known that  $(\mathcal{H}^2)^*$  is isomorphic to  $\mathcal{H}^2$  itself. By Lagrange multiplier theorem for real Banach spaces, D. Chen, T. Zheng and H.-C. Yang proved the following theorem [14]:

**Theorem 2.1.** Assume that  $\{\mu_j\}_{j=1}^\infty$  is a nondecreasing sequence, i.e.,

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \nearrow \infty,$$

where each  $\mu_i$  has finite multiplicity  $\mu_i$  and is repeated according to its multiplicity. Define

$$\begin{aligned} B &= \sum_{j=1}^{\infty} x_j^2 > 0, \\ A &= \sum_{j=1}^{\infty} \mu_j^2 x_j^2, x = (x_j)_{j=1}^\infty \in \mathcal{H}^\infty. \end{aligned} \tag{2.1}$$

If

$$x_{m_1} \neq 0$$

and

$$\sum_{j=1}^{\infty} \mu_j x_j^2 < \sqrt{AB},$$

under the conditions in (2.1), we have

$$\sum_{j=1}^{\infty} \mu_j x_j^2 \leq \frac{A + \mu_{m_1} \mu_{m_1+1} B}{\mu_{m_1} + \mu_{m_1+1}}. \tag{2.2}$$

Next, we complete the proof of the general formula by using the same method as in D. Chen, T. Zheng and H.-C. Yang [14]. For the convenience of readers, we shall give a self contained proof.

**Lemma 2.2.** Let  $(M^n, g, f)$  be an  $n$ -dimensional complete metric measure space and  $\Omega$  a bounded domain with piecewise smooth boundary  $\partial\Omega$  on the Riemannian manifold  $M^n$ . Assume that  $\lambda_i$  is the  $i^{\text{th}}$  eigenvalue of the Dirichlet problem (1.17) and  $u_i$  is an orthonormal eigenfunction corresponding to  $\lambda_i$ ,  $i = 1, 2, \dots$ , such that

$$\begin{cases} \Delta_f u_i = -\lambda_i u_i, & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u_i u_j e^{-f} dv = \delta_{ij}, & \text{for any } i, j = 1, 2, \dots \end{cases}$$

Then, for any function  $h(x) \in C^3(\Omega) \cap C^2(\overline{\Omega})$  and any integer  $k, i \in \mathbb{Z}^+$ , ( $k > i \geq 1$ ), eigenvalues of the Dirichlet problem (1.17) satisfy

$$((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) \|\nabla h u_i\|_{\Omega}^2 + \sum_{j=1}^k (\lambda_j - \lambda_i)^2 \|h u_i\|_{\Omega}^2 \leq \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{\Omega}^2, \tag{2.3}$$

where

$$\|h(x)\|_{\Omega}^2 = \int_{\Omega} h^2(x) e^{-f} dv.$$

**Proof.** Since  $u_j$  is an orthonormal eigenfunction corresponding to the eigenvalue  $\lambda_j$ ,  $\{u_j\}_{j=1}^{\infty}$  forms an orthonormal basis of the weighted  $L^2(\Omega)$ . From the Rayleigh-Ritz inequality [12], we have

$$\lambda_{k+1} \leq -\frac{\int_{\Omega} \varphi \Delta_f \varphi e^{-f} dv}{\int_{\Omega} \varphi^2 e^{-f} dv}, \quad (2.4)$$

for any function  $\varphi$  satisfies

$$\int_{\Omega} \varphi u_j e^{-f} dv = 0, \quad 1 \leq j \leq k.$$

Putting

$$a_{ij} = \int_{\Omega} h u_i u_j e^{-f} dv$$

and

$$\varphi_i = h u_i - \sum_{j=1}^k a_{ij} u_j,$$

then, we have

$$a_{ij} = a_{ji}. \quad (2.5)$$

By a simple calculation, we find that

$$\int_{\Omega} \varphi_i u_l e^{-f} dv = 0, \quad (2.6)$$

for  $1 \leq i, l \leq k$ . (2.4) implies

$$\lambda_{k+1} \leq -\frac{\int_{\Omega} \varphi_i \Delta_f \varphi_i e^{-f} dv}{\int_{\Omega} \varphi_i^2 e^{-f} dv}.$$

By defining

$$b_{ij} = - \int_{\Omega} (u_j \Delta_f h + 2 \langle \nabla h, \nabla u_j \rangle) u_i e^{-f} dv,$$

we have

$$b_{ij} = (\lambda_i - \lambda_j) a_{ij}. \quad (2.7)$$

From the Stokes' theorem, we have

$$-2 \int_{\Omega} h u_i \langle \nabla \bar{h}, \nabla u_i \rangle = - \int_{\Omega} h \langle \nabla \bar{h}, \nabla u_i^2 \rangle = \int_{\Omega} (\langle \nabla h, \nabla \bar{h} \rangle + h \Delta_f \bar{h}) u_i^2. \quad (2.8)$$

From (1.13), (2.5) and (2.7), we deduce that

$$\begin{aligned}
\int_{\Omega} \overline{\varphi}_i \Delta_f \varphi_i e^{-f} dv &= \int_{\Omega} \overline{\varphi}_i \Delta_f \left( hu_i - \sum_{j=1}^k a_{ij} u_j \right) e^{-f} dv \\
&= \int_{\Omega} \overline{\varphi}_i \left( \Delta_f(hu_i) - \Delta_f \left( \sum_{j=1}^k a_{ij} u_j \right) \right) e^{-f} dv \\
&= u_i \Delta_f h - \lambda_i hu_i + 2 \langle \nabla h, \nabla u_i \rangle + \sum_{j=1}^k \lambda_j a_{ij} u_j, \\
&= \sum_{j=k+1}^{\infty} \overline{a}_{ij} b_{ij} - \lambda_i \sum_{j=k+1}^{\infty} |a_{ij}|^2 \\
&= \frac{1}{2} \sum_{j=k+1}^{\infty} (\lambda_i - \lambda_j) |a_{ij}|^2 - \lambda_i \sum_{j=k+1}^{\infty} |a_{ij}|^2
\end{aligned} \tag{2.9}$$

From the Rayleigh-Ritz inequality (cf. [12]) and (2.9), we have

$$\lambda_{k+1} \leq - \frac{\int_{\Omega} \overline{\varphi}_i \Delta_f \varphi_i e^{-f} dv}{\int_{\Omega} |\varphi_i|^2 e^{-f} dv} = \frac{\sum_{j=k+1}^{\infty} (\lambda_i - \lambda_j) |a_{ij}|^2}{\lambda_i \sum_{j=k+1}^{\infty} |a_{ij}|^2} + \lambda_i$$

i.e.

$$(\lambda_{k+1} - \lambda_i) \sum_{j=k+1}^{\infty} |a_{ij}|^2 \leq \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i) |a_{ij}|^2. \tag{2.10}$$

Utilizing the Cauchy-Schwarz inequality, we yield

$$\left( \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i) |a_{ij}|^2 \right)^2 \leq \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i)^2 |a_{ij}|^2 \sum_{j=k+1}^{\infty} |a_{ij}|^2,$$

which is equivalent to the following:

$$\left( \|\nabla hu_i\|_{\Omega}^2 - \sum_{j=1}^k (\lambda_j - \lambda_i) |a_{ij}|^2 \right)^2 \leq \left( \|hu_i\|_{\Omega}^2 - \sum_{j=1}^k |a_{ij}|^2 \right) \left( \|2 \langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{\Omega}^2 - \sum_{j=1}^k (\lambda_j - \lambda_i)^2 |a_{ij}|^2 \right).$$

Define

$$\begin{aligned}
\mathcal{A}(i) &= \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{\Omega}^2 - \sum_{j=1}^k (\lambda_j - \lambda_i)^2 |a_{ij}|^2 \\
&= \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i)^2 |a_{ij}|^2 \geq 0; \\
\mathcal{B}(i) &= \|hu_i\|_{\Omega}^2 - \sum_{j=1}^k |a_{ij}|^2 = \sum_{j=k+1}^{\infty} |a_{ij}|^2, \quad \text{here } \int_{\Omega} hu_i u_{k+1} e^{-f} dv \neq 0;
\end{aligned}$$

and

$$C(i) = \|\nabla hu_i\|_{\Omega}^2 - \sum_{j=1}^k (\lambda_j - \lambda_i) |a_{ij}|^2 = \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i) |a_{ij}|^2.$$

Since  $hu_i$  is not the  $\mathbb{C}$ -linear combination of  $u_1, \dots, u_{k+1}$ , there exists some  $l > k + 1$  such that

$$a_l = \int_{\Omega} hu_i u_l e^{-f} dv \neq 0.$$

It is not difficult to see that

$$\lambda_i < \lambda_{k+1} < \lambda_{k+2} \leq \lambda_l,$$

therefore, the vector

$$\left(|a_{ij}|\right)_{j=k+1}^{\infty}$$

is not proportional to

$$\left((\lambda_j - \lambda_i)^2 |a_{ij}|\right)_{j=k+1}^{\infty}.$$

From the Cauchy-Schwarz inequality, we have

$$C(i) < \sqrt{\mathcal{A}(i)\mathcal{B}(i)} \tag{2.11}$$

Since  $a_{k+1} \neq 0$ , from (2.11) and theorem 2.1, we obtain

$$C(i) \leq \frac{\mathcal{A}(i) + (\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)\mathcal{B}(i)}{(\lambda_{k+2} - \lambda_i) - (\lambda_{k+1} - \lambda_i)} \tag{2.12}$$

From (2.12), and the definition of  $\mathcal{A}(i)$ ,  $\mathcal{B}(i)$  and  $C(i)$ , one can infer that

$$\begin{aligned}
&((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i))\|\nabla hu_i\|_{\Omega}^2 \\
&\leq \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{\Omega}^2 - \sum_{j=1}^k (\lambda_j - \lambda_i)^2 \|hu_i\|_{\Omega}^2 - (\lambda_{k+2} - \lambda_j)(\lambda_{k+1} - \lambda_j) |a_{ij}|^2 \\
&\leq \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{\Omega}^2 - \sum_{j=1}^k (\lambda_j - \lambda_i)^2 \|hu_i\|_{\Omega}^2.
\end{aligned}$$

Therefore, we complete the proof of this lemma.  $\square$

**Corollary 2.3.** *Under the assumption of the lemma 2.2, for any real value function  $F \in C^3(\Omega) \cap C^2(\overline{\Omega})$ , we have*

$$((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) \|\nabla F u_i\|_{\Omega}^2 \leq 2 \sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)} \|\nabla F\|^2 u_i\|_{\Omega}^2 + \|2\langle \nabla F, \nabla u_i \rangle + u_i \Delta_f F\|_{\Omega}^2. \quad (2.13)$$

**Proof.** Taking  $h = e^{i\alpha F}$ ,  $F \in \mathbb{R} \setminus \{0\}$  in (2.3), we obtain

$$\eta^2((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) \|\nabla F u_i\|_{\Omega}^2 \leq \eta^4 \|\nabla F\|^2 u_i\|_{\Omega}^2 + \eta^2 \|2\langle \nabla F, \nabla u_i \rangle + u_i \Delta_f F\|_{\Omega}^2 + (\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i). \quad (2.14)$$

From (2.14), we deduce

$$((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) \|\nabla F u_i\|_{\Omega}^2 \leq \eta^2 \|\nabla F\|^2 u_i\|_{\Omega}^2 + \|2\langle \nabla F, \nabla u_i \rangle + u_i \Delta_f F\|_{\Omega}^2 + \frac{1}{\eta^2} (\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i). \quad (2.15)$$

Using the Cauchy-Schwarz inequality in (2.15), we yield (2.16). This finishes the proof of this lemma.  $\square$

By utilizing corollary 2.3, we have

**Proposition 2.4.** *Let  $\tau$  be a constant such that, for any  $i = 1, 2, \dots, k$ ,  $\lambda_i + \tau > 0$ . Under the assumption of the lemma 2.2, for any  $j = 1, 2, \dots, l$ , and any real value function  $F_j \in C^3(\Omega) \cap C^2(\overline{\Omega})$ , we have*

$$\sum_{j=1}^l \frac{a_j^2 + b_j}{2} (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \tau) \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{\Omega}^2, \quad (2.16)$$

where

$$\begin{aligned} a_j &= \sqrt{\|\nabla F_j u_i\|_{\Omega}^2}, \\ b_j &= \sqrt{\|\nabla F_j\|^2 u_i\|_{\Omega}^2}, \\ a_j^2 &\geq b_j, \end{aligned} \quad (2.17)$$

and

$$\|F(x)\| = \int_{\Omega} F(x) e^{-f} dv.$$

**Proof.** By the assumption, we have

$$\frac{a_j^2 - b_j}{2} \left( \sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \geq 0$$

which is equivalent to the following:

$$\begin{aligned} & a_j^2((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)) - 2b_j \sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)} \\ & \geq \frac{a_j^2 + b_j}{2} \left( \sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i} \right)^2. \end{aligned} \quad (2.18)$$

By (2.18) and (2.3), we have

$$\frac{a_j^2 + b_j}{2} \left( \sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \leq \|2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j\|_{\Omega}^2.$$

Taking sum over  $j$  from 1 to  $l$ , we yield

$$\sum_{j=1}^l \frac{a_j^2 + b_j}{2} \left( \sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \leq \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta F_j\|_{\Omega}^2. \quad (2.19)$$

Multiplying (2.19) by  $\left( \sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i} \right)^2$  on both sides, one can infer that

$$\begin{aligned} \sum_{j=1}^l \frac{a_j^2 + b_j}{2} (\lambda_{k+2} - \lambda_{k+1})^2 & \leq \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta F_j\|_{\Omega}^2 \left( \sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i} \right)^2 \\ & = \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta F_j\|_{\Omega}^2 \\ & \quad \times \left( \sqrt{(\lambda_{k+2} + \tau) - (\lambda_i + \tau)} + \sqrt{(\lambda_{k+1} + \tau) - (\lambda_i + \tau)} \right)^2 \\ & \leq 4(\lambda_{k+2} + \tau) \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta F_j\|_{\Omega}^2. \end{aligned}$$

which is the inequality (2.16). Therefore, we finish the proof of this proposition.  $\square$

By the same method as the proof of proposition 2.4, one can prove the following proposition if one notices to count the number of eigenvalues from 0.

**Proposition 2.5.** *Let  $(M^n, g)$  be an  $n$ -dimensional closed metric measure space. Assume that  $\bar{\lambda}_i$  is the  $i^{\text{th}}$  eigenvalue of the eigenvalue problem (1.8) and  $u_i$  is an orthonormal eigenfunction corresponding*

to  $\bar{\lambda}_i$ ,  $i = 0, 1, 2, \dots$ , such that

$$\begin{cases} \Delta_f u_i = -\lambda u_i, & \text{in } M^n, \\ \int_{M^n} u_i u_j e^{-f} dv = \delta_{ij}, & \text{for any } i, j = 0, 1, 2, \dots \end{cases}$$

Let  $\tau$  be a constant such that, for any  $i = 0, 1, 2, \dots, k$ ,  $\bar{\lambda}_i + \tau > 0$ . Then, for any  $j = 0, 1, 2, \dots, l$ , and any real value function  $h_j \in C^2(M^n)$ , we have

$$\sum_{j=1}^l \frac{a_j^2 + b_j}{2} (\bar{\lambda}_{k+2} - \bar{\lambda}_{k+1})^2 \leq 4(\bar{\lambda}_{k+2} + \tau) \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{M^n}^2, \quad (2.20)$$

where

$$a_j = \sqrt{\|\nabla F_j u_i\|_{M^n}^2},$$

$$b_j = \sqrt{\| |\nabla F_j|^2 u_i \|_{M^n}^2},$$

$$\|h\|^2 \int_{M^n} h^2 e^{-f} dv,$$

and

$$a_j^2 \geq b_j.$$

Next, we state the general formula given by C. Xia and H. Xu in [81], which will be used in the next section.

**Proposition 2.6.** *Let  $(M^n, g)$  be an  $n$ -dimensional complete metric measure space. Assume that  $\lambda_i$  is the  $i^{\text{th}}$  eigenvalue of the Dirichlet problem (1.17) and  $u_i$  is an orthonormal eigenfunction corresponding to  $\lambda_i$ ,  $i = 1, 2, \dots$ , such that*

$$\Delta_f u_i = -\lambda_i u_i \text{ and } \int_{\Omega} u_i u_j e^{-f} dv = \delta_{ij}, \text{ for any } i, j = 1, 2, \dots$$

Then, for any function  $h(x) \in C^2(\Omega)$  and any positive integer  $k$ , eigenvalues of the Dirichlet problem (1.17) satisfy

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \|u_i \nabla h\|_{\Omega}^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{\Omega}^2. \quad (2.21)$$

Similarly, Q.-M. Cheng and the author [20] proved the following (also see [84]):



**Proposition 2.7.** *Let  $(M^n, g, \cdot)$  be an  $n$ -dimensional closed metric measure space. Assume that  $\bar{\lambda}_i$  is the  $i^{\text{th}}$  eigenvalue of the close eigenvalue problem (1.8) and  $u_i$  is an orthonormal eigenfunction corresponding to  $\bar{\lambda}_i$ ,  $i = 0, 1, 2, \dots$ , such that*

$$\Delta_f u_i = -\bar{\lambda}_i u_i \text{ and } \int_{M^n} u_i u_j e^{-f} dv = \delta_{ij}, \text{ for any } i, j = 0, 1, 2, \dots.$$

*Then, for any function  $h(x) \in C^2(M^n)$  and any positive integer  $k$ , eigenvalues of the close eigenvalue problem (1.8) satisfy*

$$\sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i)^2 \|u_i \nabla h\|_{M^n}^2 \leq \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i) \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta_f h\|_{M^n}^2. \quad (2.22)$$

*Remark 2.1.* However, it is very well known that the drifting Laplacian  $\Delta f := \Delta + \langle \nabla f, \cdot \rangle$  on a compact measure metric space  $(M^n, g, e^{-f} dv)$  is unitarily equivalent to the Schrödinger operator

$$\Delta + \frac{1}{2} \Delta f + \frac{1}{4} |\nabla f|^2$$

on  $(M^n, g)$  and thus it has the same spectrum (see for instance [71]). Therefore, proposition 2.7 can be proved by the similar method in [42, 43, 72, 73, 76] by replacing the potential  $q$  or  $V$  in that papers by  $\Delta + \frac{1}{2} \Delta f + \frac{1}{4} |\nabla f|^2$ .

### 3 Proofs of theorem 1.2 and theorem 1.3

In this section, we would like to give the proofs of theorem 1.2 and theorem 1.3. Firstly, we need the following lemma which will be found in [13].

**Lemma 3.1.** *For an  $n$ -dimensional submanifold  $M^n$  in Euclidean space  $\mathbb{R}^{n+p}$ , let  $y = (y^1, y^2, \dots, y^{n+p})$  is the position vector of a point  $p \in M^n$  with  $y^\alpha = y^\alpha(x_1, \dots, x_n)$ ,  $1 \leq \alpha \leq n+p$ , where  $(x_1, \dots, x_n)$  denotes a local coordinate system of  $M^n$ . Then, we have*

$$\sum_{\alpha=1}^{n+p} g(\nabla y^\alpha, \nabla y^\alpha) = n, \quad (3.1)$$

$$\sum_{\alpha=1}^{n+p} g(\nabla y^\alpha, \nabla u) g(\nabla y^\alpha, \nabla w) = g(\nabla u, \nabla w), \quad (3.2)$$

for any functions  $u, w \in C^1(M^n)$ ,

$$\sum_{\alpha=1}^{n+p} (\Delta y^\alpha)^2 = n^2 H^2, \quad (3.3)$$

$$\sum_{\alpha=1}^{n+p} \Delta y^\alpha \nabla y^\alpha = 0, \quad (3.4)$$

where  $H$  is the mean curvature of  $M^n$ .

Utilizing the general formula (2.21), one can deduce that

$$\begin{aligned} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 &\leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left( \lambda_i + \int_{\Omega} (n^2 H^2 + 2\Delta f - |\nabla f|^2) e^{-f} dv \right) \\ &\leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (\lambda_i + c), \end{aligned} \quad (3.5)$$

where

$$c := \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} (n^2 H^2 + 2|\Delta_\psi f| + |\nabla f|^2), \quad (3.6)$$

and  $\Psi$  denotes the set of all isometric immersions from  $M^n$  into a Euclidean space.

**A recursion formula of Cheng and Yang.** Let  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{k+1}$  be any positive real numbers satisfying

$$\sum_{i=1}^k (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n} \sum_{i=1}^k \mu_i (\mu_{k+1} - \mu_i).$$

Define

$$\Lambda_k = \frac{1}{k} \sum_{i=1}^k \mu_i, \quad T_k = \frac{1}{k} \sum_{i=1}^k \mu_i^2, \quad F_k = \left(1 + \frac{2}{n}\right) \Lambda_k^2 - T_k.$$

Then, we have

$$F_{k+1} \leq C(n, k) \left( \frac{k+1}{k} \right)^{\frac{4}{n}} F_k, \quad (3.7)$$

where

$$C(n, k) = 1 - \frac{1}{3n} \left( \frac{k}{k+1} \right)^{\frac{4}{n}} \frac{\left(1 + \frac{2}{n}\right) \left(1 + \frac{4}{n}\right)}{(k+1)^3} < 1.$$

By using the recursion formula given by Q.-M. Cheng and H.-C. Yang, the author proved the following

**Theorem 3.2.** *Let  $(M^n, g, f)$  be an  $n$ -dimensional complete metric measure space.  $\lambda_k$  denotes the  $k$ -th eigenvalue of the Dirichlet problem (1.17) of the drifting Laplacian. Then, for any  $k \geq 1$ ,*

$$\lambda_{k+1} + c \leq \left(1 + \frac{4}{n}\right) (\lambda_1 + c) k^{2/n}, \quad (3.8)$$

where  $c$  is the same constant as in (3.6).

**Proof of theorem 1.2.** Let  $F_j(x) = \alpha_j x^j$  and  $a_j > 0$ , such that

$$a_j^2 = \|\nabla F_j u_i\|_\Omega^2 \geq \sqrt{\|\nabla F_j\|^2 u_i\|_\Omega^2} = b_j \geq 0,$$

$$\sum_{j=1}^{n+p} \int 2u_i \langle \nabla F_j, \nabla f \rangle \Delta F_j e^{-f} dv = 0, \quad (3.9)$$

and

$$\sum_{j=1}^{n+p} \int 2u_i \langle \nabla F_j, \nabla u_i \rangle \Delta F_j e^{-f} dv = 0, \quad (3.10)$$

where  $j = 1, 2, \dots, n+p$ , and  $x^j$  denotes the  $j$ -th standard coordinate function of the Euclidean space  $\mathbb{R}^{n+p}$ . Let

$$\alpha = \min_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\bar{\alpha} = \max_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\beta = \min_{1 \leq j \leq n+p} \{b_j\},$$

and  $l = n + p$ , then, by lemma 2.2 and (3.1), we have

$$\begin{aligned} \sum_{j=1}^l \frac{a_j^2 + b_j}{2} &= \sum_{j=1}^{n+p} \frac{a_j^2 + b_j}{2} \\ &\geq \frac{1}{2} \left( n\alpha^2 + \sum_{j=1}^{n+p} b_j \right) \\ &\geq \frac{1}{2} (n\alpha^2 + (n+p)\beta), \end{aligned} \quad (3.11)$$

and

$$\sum_{j=1}^{n+p} \int_\Omega u_i^2 \langle \nabla F_j, \nabla f \rangle^2 e^{-f} dv = \sum_{j=1}^{n+p} \int_\Omega u_i^2 \langle \nabla (a_j x^j), \nabla f \rangle^2 e^{-f} dv \leq \bar{\alpha}^2 \int_\Omega u_i^2 |\nabla f|^2 e^{-f} dv. \quad (3.12)$$

From (3.3), we obtain

$$\sum_{j=1}^{n+p} (\Delta F_j)^2 = \sum_{j=1}^{n+p} (\Delta (a_j x^j))^2 \leq \bar{\alpha}^2 n^2 H^2. \quad (3.13)$$

Utilizing (3.2) and (1.13), we have

$$\begin{aligned}
-4 \sum_{j=1}^{n+p} \int_{\Omega} u_i \langle \nabla F_j, \nabla u_i \rangle \langle \nabla F_j, \nabla f \rangle e^{-f} dv &= -4 \sum_{j=1}^{n+p} \int_{\Omega} u_i \langle \nabla (a_j x^j), \nabla u_i \rangle \langle \nabla (a_j x^j), \nabla f \rangle e^{-f} dv \\
&\leq 2\bar{\alpha}^2 \left| \int_{\Omega} \langle \nabla f, \nabla u_i^2 \rangle e^{-f} dv \right| \\
&= 2\bar{\alpha}^2 \left| \int_{\Omega} u_i^2 \Delta_f f e^{-f} dv \right|,
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\sum_{j=1}^{n+p} \int_{\Omega} \langle \nabla F_j, \nabla u_i \rangle^2 e^{-f} dv &= \sum_{j=1}^{n+p} \int_{\Omega} \langle \nabla (a_j x^j), \nabla u_i \rangle^2 e^{-f} dv \\
&\leq \bar{\alpha}^2 \sum_{j=1}^{n+p} \int_{\Omega} \langle \nabla x_j, \nabla u_i \rangle^2 e^{-f} dv \\
&= \bar{\alpha}^2 \lambda_i.
\end{aligned} \tag{3.15}$$

Let  $\Psi$  denote the set of all isometric immersions from  $M^n$  into a Euclidean space. Define

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (|\nabla f|^2 + 2|\Delta_f f| + n^2 H^2) > 0.$$

Since eigenvalues are invariant under isometries, by lemma 3.1, (3.9), (3.10), and (3.12)-(3.15), we have

$$\begin{aligned}
&4(\lambda_{k+2} + c) \sum_{j=1}^{n+p} \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{\Omega}^2 \\
&\leq 4(\lambda_{k+2} + c) \bar{\alpha}^2 \left( 4\lambda_i + \int_{\Omega} u_i^2 |\nabla f|^2 e^{-f} dv + 2 \left| \int_{\Omega} u_i^2 \Delta_f f e^{-f} dv \right| + \int_{\Omega} u_i^2 n^2 H^2 e^{-f} dv \right) \\
&\leq 4(\lambda_{k+2} + c) \bar{\alpha}^2 \left( 4\lambda_i + \int_{\Omega} u_i^2 (|\nabla f|^2 + 2|\Delta_f f| + n^2 H^2) e^{-f} dv \right) \\
&\leq 16\lambda_{k+2} \bar{\alpha}^2 (\lambda_i + c).
\end{aligned} \tag{3.16}$$

In proposition 2.4, we let  $i = 1, \tau = c$ . Then, from (2.16), we have

$$(\lambda_{k+2} - \lambda_{k+1})^2 \leq \frac{32\bar{\alpha}^2(\lambda_{k+2} + c)}{n\alpha^2 + (N)\beta} (\lambda_1 + c). \tag{3.17}$$

Furthermore, we deduce from (3.17) and (3.8) that,

$$\begin{aligned}\lambda_{k+2} - \lambda_{k+1} &\leq \sqrt{\frac{32\bar{\alpha}^2}{n\alpha^2 + (n+p)\beta}} \sqrt{\lambda_1 + c} \sqrt{\lambda_{k+2} + c} \\ &\leq (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}} (k+1)^{\frac{1}{n}} \\ &= C_{n,\Omega,f} (k+1)^{\frac{1}{n}},\end{aligned}$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}}.$$

Therefore, we finish the proof of theorem 1.2.

□

According to the proof of theorem 1.2, it is not difficult to obtain the following corollary:

**Corollary 3.3.** *Let  $(M^n, g, f)$  be an  $n$ -dimensional complete metric measure space isometrically immersed in a Euclidean space  $\mathbb{R}^{n+p}$ , and  $\lambda_i$  be the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the Dirichlet problem (1.17). Then, for any  $k = 1, 2, \dots$ , there are  $(n+p+1)$  constants  $\alpha$ , and  $b_j, j = 1, 2, \dots, n+p$ , such that*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (3.18)$$

where

$$\begin{aligned}C_{n,\Omega,f} &= (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}}, \\ c &= \frac{1}{4} \int_{\Omega} u_i^2 (|\nabla f|^2 + 2|\Delta_f f| + n^2 H^2) e^{-f} dv,\end{aligned}$$

and  $C_0(n)$  is the same as the one in (1.6).

*Remark 3.1.* If  $M^n$  is an  $n$ -dimensional Euclidean space, then we have  $H = 0$ , and thus  $c = 0$ . Let  $\alpha_j = 1$ , where  $j = 1, 2, \dots, n+p$ , then  $h_j = x^j$ . Thus, we have

$$\alpha = 1,$$

and

$$\sum_{j=1}^{n+p} b_j = n,$$

which implies that

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}} = 4\lambda_1 \sqrt{\frac{C_0(n)}{n}}.$$

By (3.4), we know that the assumption (3.10) holds for the above case. Therefore, our result is sharper than Chen-Zheng-Yang's eigenvalue inequality (4.26).

*Remark 3.2.* In corollary 3.3, we assume that the complete Riemannian manifold  $M^n$  is a minimal submanifold of the Euclidean space  $\mathbb{R}^{n+p}$ , then the constant  $c$  is given by

$$c = \frac{1}{4} \int_{M^n} u_i^2 (|\nabla f|^2 + 2|\Delta_f f|) e^{-f} dv.$$

Furthermore, if  $f$  is a constant, it is clear that the constant  $c = 0$ .

**Proof of theorem 1.3.** By proposition 2.16 and lemma 3.1, we can give the proof by using the same method as the proof of theorem 1.2. □

Similarly, we have the following:

**Corollary 3.4.** Let  $(M^n, g, f)$  be an  $n$ -dimensional closed metric measure space, and  $\bar{\lambda}_i$  be the  $i$ -th ( $i = 0, 1, 2, \dots, k$ ) eigenvalue of the closed eigenvalue problem (1.8). Then, for any  $k = 1, 2, \dots$ , there are  $(n + p + 1)$  constants  $\alpha$ , and  $b_j, j = 1, 2, \dots, n + p$ , such that

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq C_{n,M^n,f} k^{\frac{1}{n}},$$

where

$$C_{n,M^n,f} = (\bar{\lambda}_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}},$$

and  $C_0(n)$  is the same as the one in (1.6), and

$$c = \frac{1}{4} \int_{M^n} u_i^2 (|\nabla f|^2 + 2|\Delta_f f| + n^2 H^2) e^{-f} dv.$$

*Remark 3.3.* In corollary 3.4, we assume that  $M^n$  is a minimal submanifold of the Euclidean space  $\mathbb{R}^{n+p}$ , then the constant  $c$  is given by

$$c = \frac{1}{4} \int_{M^n} u_i^2 (|\nabla f|^2 + 2|\Delta_f f|) e^{-f} dv.$$

Furthermore, if  $f$  is a constant, then we know that the constant  $c = 0$ .

In theorem 1.2, the best constant  $C_{n,\Omega,f}$  is called the gap coefficient. We shall note that it is worth noting that it is very difficult for us to give the explicit expression of the optimal gap coefficient, even if  $\Omega$  are some special domains in the Euclidean space with dimension  $n$  and  $f$  is a constant. Let  $\Omega$  be a bounded domain with piecewise smooth boundary  $\partial\Omega$  on an  $n$ -dimensional Riemannian manifold  $M^n$ . If  $\lambda_i$  is the  $i$ -th eigenvalue of Dirichlet problem (1.1). According to a great amount of numeric calculation for some special examples, the author conjectured that [87]: for any positive integer  $k$ ,

$$\lambda_{k+1} - \lambda_k \leq (\lambda_2 - \lambda_1)k^{\frac{1}{n}}.$$

Therefore, in the sense of metric measure space, it is natural to generalize the above conjecture to the following:

**Conjecture 3.5.** *Let  $\Omega$  be a bounded domain with piecewise smooth boundary  $\partial\Omega$  on an  $n$ -dimensional Riemannian manifold  $M^n$ . If  $\lambda_i$  is the  $i$ -th eigenvalue of Dirichlet problem (1.17) of the drifting Laplacian. Then, for any positive integer  $k$ ,*

$$\lambda_{k+1} - \lambda_k \leq (\lambda_2 - \lambda_1)k^{\frac{1}{n}}. \quad (3.19)$$

*Remark 3.4.* As we know, for the Dirichlet problem (1.17) on the Riemannian manifolds, the gap of the consecutive eigenvalues  $\lambda_{k+1} - \lambda_k$  is bounded by the first  $k$ -th eigenvalues in the previous literatures. However, from the above conjecture, we know that the gap of the consecutive eigenvalues is bounded only by the first two eigenvalues.

## 4 Eigenvalues on the Ricci Solitons

As an application of general formula of eigenvalues of drifting Laplacian on complete metric measure spaces, we will consider the gradient Ricci solitons in this section. Recall that Ricci solitons play an important role as singularity dilations in the Ricci flow proof of uniformization, see [22]. They correspond to self-similar solutions of Ricci flow [31], and usually serve as natural generalizations of Einstein metrics. Assume that  $S$  denotes the scalar curvature of  $M^n$ , then we have the following

**Proposition 4.1.** *For an  $n$ -dimensional closed gradient Ricci soliton  $(M^n, g, f)$ , for any  $k$ , eigenvalues of the closed eigenvalue problem (1.8) of drifting Laplacian satisfy*

$$\sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i)^2 \leq \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i)(\bar{\lambda}_i + c), \quad (4.1)$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} (n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S),$$

and

$$\bar{c} = \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}.$$

**Proof.** By making use of equation (1.21), we have (cf. [9, 27]):

$$S + \Delta f = n\rho, \quad (4.2)$$

and

$$S + |\nabla f|^2 = 2\rho f + \bar{c}, \quad (4.3)$$

where  $S$  denotes the scalar curvature of  $M^n$  and  $\bar{c}$  is a constant. From (4.2) and (4.3), we have

$$\Delta_f f = n\rho - 2\rho f - \bar{c}. \quad (4.4)$$

Therefore, by integrating for (4.4), we obtain

$$\bar{c} = n\rho - 2\rho \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}. \quad (4.5)$$

By making use of (4.2), (4.3) and (4.5), we have

$$2|\Delta_f f| + |\nabla f|^2 = |2n\rho - 4\rho f - 2\bar{c}| + 2\rho f + \bar{c} - S. \quad (4.6)$$

Hence, from (4.6), we obtain

$$\begin{aligned} & \int_{M^n} u_i^2 (2|\Delta_f f| + |\nabla f|^2) e^{-f} dv \\ &= \int_{M^n} u_i^2 (|2n\rho - 4\rho f - 2\bar{c}| + 2\rho f + \bar{c} - S) e^{-f} dv \\ &= \int_{M^n} u_i^2 (4|\rho f - \rho\bar{c}| + 2\rho f + n\rho - 2\rho\bar{c} - S) e^{-f} dv, \end{aligned} \quad (4.7)$$

where

$$\bar{c} = \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}.$$



Recall that, in [20], Cheng and the author proved the following(also see [84]):

$$\begin{aligned} & \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i)^2 \\ & \leq \frac{4}{n} \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i) \left( \bar{\lambda}_i + \frac{1}{4} \int_{M^n} u_i^2 (n^2 H^2 + 2\Delta f - |\nabla f|^2) e^{-f} dv \right). \end{aligned} \quad (4.8)$$

Therefore, it follows from (4.8) that ,

$$\begin{aligned} & \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i)^2 \\ & \leq \frac{4}{n} \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i) \left( \bar{\lambda}_i + \frac{1}{4} \int_{M^n} u_i^2 (n^2 H^2 + 2|\Delta_f f| + |\nabla f|^2) e^{-f} dv \right) \\ & \leq \frac{4}{n} \sum_{i=0}^k (\bar{\lambda}_{k+1} - \bar{\lambda}_i) (\bar{\lambda}_i + c), \end{aligned} \quad (4.9)$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} \left( n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S \right).$$

Therefore, we finish the proof of this proposition.  $\square$

**Proposition 4.2.** Assume that  $M^n$  is a submanifold in the Euclidean space  $\mathbb{R}^{n+p}$  and  $H$  is the mean curvature of the submanifold  $M^n$ . For an  $n$ -dimensional complete gradient Ricci soliton  $(M^n, g, f)$ , there exists a function  $H$  such that, for any  $k$ , eigenvalues of the Dirichlet problem (1.17) of drifting Laplacian satisfy

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (\lambda_i + c), \quad (4.10)$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} \left( n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S \right),$$

and

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

**Lemma 4.3.** For an  $n$ -dimensional closed Ricci soliton  $(M^n, g, f)$ , the  $k^{th}$  eigenvalue  $\lambda_k$  of the eigenvalue problem (1.8) of the drifting Laplacian satisfy, for any  $k \geq 1$ ,

$$\bar{\lambda}_{k+1} + c \leq \left(1 + \frac{4}{n}\right) (\bar{\lambda}_1 + c) k^{2/n},$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} \left( n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S \right),$$

and

$$\bar{c} = \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}.$$

**Proof.** Putting

$$\mu_{i+1} = \lambda_i + c > 0,$$

for any  $i = 0, 1, 2, \dots$ . Then, we obtain from (4.10)

$$\sum_{i=1}^k (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\mu_{k+1} - \mu_i) \mu_i. \quad (4.11)$$

By making use of the same proof as in Cheng and Yang [19], we can complete our proof of lemma 4.3.  $\square$

Applying proposition 4.2 and lemma 4.3, we shall give the proof of theorem 1.4.

**Proof of theorem 1.4.** Let  $F_j(x) = \alpha_j x^j$  and  $a_j > 0$ , such that

$$a_j^2 = \|\nabla F_j u_i\|_{M^n}^2 = \sqrt{\|\nabla F_j\|^2 \|u_i\|_{M^n}^2} = b_j \geq 0,$$

$$\sum_{j=1}^{n+p} \int 2u_i \langle \nabla F_j, \nabla f \rangle \Delta F_j e^{-f} dv = 0,$$

and

$$\sum_{j=1}^{n+p} \int 2u_i \langle \nabla F_j, \nabla u_i \rangle \Delta F_j e^{-f} dv = 0,$$

where  $j = 1, 2, \dots, n+p$ , and  $x^j$  denotes the  $j$ -th standard coordinate function of the Euclidean space  $\mathbb{R}^N$ . Let

$$\alpha = \min_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\bar{\alpha} = \max_{1 \leq j \leq N} \{\alpha_j\},$$

$$\beta = \min_{1 \leq j \leq N} \{b_j\}.$$

Then, we have

$$\sum_{j=1}^l \frac{a_j^2 + b_j}{2} \geq \frac{1}{2} (n\alpha^2 + (n+p)\beta). \quad (4.12)$$

By the same argument as the proof of theorem 1.2, we deduce

$$\begin{aligned} & 4(\bar{\lambda}_{k+2} + c) \sum_{j=1}^{n+p} \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{M^n}^2 \\ & \leq 4(\bar{\lambda}_{k+2} + c) \bar{\alpha}^2 \left( 4\bar{\lambda}_i + \int_{M^n} u_i^2 (|\nabla f|^2 + 2|\Delta_f f| + n^2 H^2) e^{-f} dv \right) \\ & \leq 16(\bar{\lambda}_{k+2} + c) \bar{\alpha}^2 \left( \bar{\lambda}_i + \frac{1}{4} \left( \int_{M^n} u_i^2 (|\nabla f|^2 + 2|\Delta_f f|) e^{-f} dv + \int_{M^n} u_i^2 n^2 H^2 e^{-f} dv \right) \right). \end{aligned} \quad (4.13)$$

From the proof of proposition 4.2 and inequality (4.13), we infer that

$$4(\bar{\lambda}_{k+2} + c) \sum_{j=1}^N \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{M^n}^2 \leq 16\lambda_{k+2} \bar{\alpha}^2 (\bar{\lambda}_i + c), \quad (4.14)$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} (n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S),$$

and

$$\bar{c} = \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}.$$

Let  $\tau = c$  in proposition 2.5. Then, substituting (4.12) and (4.14) into (2.20), we obtain

$$\frac{1}{2} (n\alpha^2 + (n+p)\beta) (\bar{\lambda}_{k+2} - \bar{\lambda}_{k+1})^2 \leq 16\lambda_{k+2} \bar{\alpha}^2 (\bar{\lambda}_i + c),$$

which implies that

$$\begin{aligned} \bar{\lambda}_{k+2} - \bar{\lambda}_{k+1} & \leq \sqrt{\frac{32\bar{\alpha}^2}{n\alpha^2 + (n+p)\beta}} \sqrt{\lambda_1 + c} \sqrt{\lambda_{k+2} + c} \\ & \leq (\bar{\lambda}_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}} (k+1)^{\frac{1}{n}} \\ & = C_{n,M^n,f} (k+1)^{\frac{1}{n}}, \end{aligned}$$

where

$$C_{n,M^n,f} = (\bar{\lambda}_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}},$$

and  $C_0(n)$  is the same as the one in (1.6). Therefore, we finish the proof of this theorem.

□

*Remark 4.1.* For a complete gradient Ricci soliton  $M^n, g, f$ , if it is a minimal submanifold of  $\mathbb{R}^{n+p}$ , the constant  $c$  in the theorem 1.4 will be given by

$$c = \frac{1}{4} \left( n\rho + 2\rho\bar{c} - \max_{M^n} (2\rho f + R) \right),$$

and

$$\bar{c} = n\rho - 2\rho \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

*Remark 4.2.* The constant  $c$ , which is appeared in [20], is given by

$$c = \frac{1}{4} \left( n\rho + 2\rho\bar{c} + \inf_{\psi \in \Psi} \max_{M^n} (n^2 H^2 - 2\rho f - R) \right),$$

and

$$\bar{c} = \frac{\int_{M^n} f e^{-f} dv}{\int_{M^n} e^{-f} dv}.$$

This is because it is not necessary for us to compute the value of  $2\Delta f - |\nabla f|^2$  but  $2|\Delta_f f| + |\nabla f|^2$  in proposition 4.2.

*Remark 4.3.* Assume that  $(M^n, g_{ij}, f)$  is a compact, expanding or steady, gradient Ricci soliton, then, the gradient Ricci solitons is Einstein [33, 44], which means that

$$2|\Delta_f f| + |\nabla f|^2 = 0.$$

Hence, the constant  $c$  in the theorem 1.4 can be given by

$$c = \frac{1}{4} \inf_{\psi \in \Psi} n^2 H^2.$$

*Remark 4.4.* We suppose  $(M, g)$  is a Sasakian manifold satisfying the gradient Ricci soliton equation, and then  $f$  is a constant function. Therefore,  $(M, g)$  is an Einstein manifold [39], which implies that there dose not exist the compact non-Einstein Ricci soliton in Sasakian manifolds since all compact Ricci solitons are gradient ones from [65]. For this case, the constant  $c$  in the theorem 1.4 can be given by

$$c = \frac{1}{4} \inf_{\psi \in \Psi} n^2 H^2.$$

*Remark 4.5.* For a compact shrinking Ricci soliton  $(M^n, g_{ij}, f)$  with dimension  $n \leq 3$ , the gradient Ricci Solitons is Einstein [33, 44]. Hence, the constant  $c$  in the theorem 1.4 can be given by

$$c = \frac{1}{4} \inf_{\psi \in \Psi} n^2 H^2.$$

*Remark 4.6.* If  $(M^n, g, f)$  is a compact Ricci soliton with rigidity, then the constant  $c$  in the theorem 1.4 can be given by

$$c = \frac{1}{4} \inf_{\psi \in \Psi} n^2 H^2.$$

Indeed, since  $(M^n, g, f)$  is a compact Ricci soliton with rigidity, then it is a trivial Ricci soliton which means that  $f$  is a constant [56]. Therefore, we have

$$2|\Delta_f f| + |\nabla f|^2 = 0.$$

**Theorem 4.4.** *Let  $(M^n, g_{ij}, f)$  be an  $n$ -dimensional complete gradient Ricci Soliton. Then, for any  $k$ , eigenvalues of the Dirichlet problem (1.17) of the drifting Laplacian satisfy*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (4.15)$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}}, \quad (4.16)$$

$C_0(n)$  is the same as the one in (1.6),

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} \left( n^2 H^2 + 4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S \right),$$

and

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

**Proof.** The proof is similar to the theorem 1.4. Thus, we omit it here.  $\square$

*Remark 4.7.* In theorem 4.4, we further assume that  $(M^n, g)$  is an  $n$ -dimensional complete minimal submanifold of the  $(n + p)$ -dimensional Euclidean space  $\mathbb{R}^{n+p}$ . Then, the mean curvature is zero and thus it is not difficult to see that the constant  $c$  in theorem 4.4 will be given by

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S),$$

where

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

*Remark 4.8.* If we assume that  $(M^n, g, f)$  is a steady Ricci soliton, then the constant  $c$  in theorem 4.4 is given by

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{M^n} (n^2 H^2 - S).$$

**Theorem 4.5.** [59] *Let  $(M^n, g)$  be an  $n$ -dimensional complete noncompact gradient shrinking Ricci soliton whose curvature tensor has at most exponential growth and having Ricci tensor bounded from below. Then, for any  $k, k = 1, 2, \dots$ , eigenvalues of the Dirichlet problem (1.17) of the drifting Laplacian satisfy*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (4.17)$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}}, \quad (4.18)$$

$C_0(n)$  is the same as the one in (1.6),

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c} - S),$$

where

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

Let  $(M^n, g)$  be an  $n$ -dimensional, complete manifold. Suppose that there exists a smooth function  $f : M \rightarrow \mathbb{R}$  satisfying  $\text{Hess} f = \rho g$ , for some constant  $\rho \neq 0$ . Then Riemannian manifold  $M^n$  is isometric to  $\mathbb{R}^n$ . Hence, we have

**Corollary 4.6.** *Let  $(M^n, g, f)$  be an  $n$ -dimensional, complete gradient Ricci soliton with  $\text{Hess} f = \rho g$ . Assume that  $\lambda_i$  denotes the  $i$ -th eigenvalue of Dirichlet problem (1.17) of the drifting Laplacian. Then, for any  $k = 1, 2, \dots$ , we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (4.19)$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}}, \quad (4.20)$$

$C_0(n)$  is the same as the one in (1.6), where

$$c = \frac{1}{4} \left( n\rho + 2\rho\bar{c} - 2 \min_{\Omega} \rho f \right),$$

and

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

**Proof.** If  $\text{Hess} f = \rho g$ , then we have

$$\Delta f = n\rho, \quad (4.21)$$

and

$$|\nabla f|^2 = 2\rho f + \bar{c}, \quad (4.22)$$

where  $\bar{c}$  is a constant defined by

$$\bar{c} = n\rho - 2\rho \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

It follows from (4.21) and (4.22) that

$$\begin{aligned} & \int_{\Omega} u_i^2 (2|\Delta f| + |\nabla f|^2) e^{-f} dv \\ &= 2 \int_{\Omega} u_i^2 \left| 2\rho \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv} - 2\rho f \right| e^{-f} dv + 2\rho \int_{\Omega} u_i^2 f e^{-f} dv + n\rho - 2\rho \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv} \\ &= 4 \int_{\Omega} u_i^2 |\rho\bar{c} - \rho f| e^{-f} dv + 2\rho \int_{\Omega} u_i^2 f e^{-f} dv + n\rho - 2\rho\bar{c}, \end{aligned} \quad (4.23)$$

where

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

Hence, by the same method as the proof of proposition 4.2, we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (\lambda_i + c),$$

where

$$c = \frac{1}{4} \min_{\psi \in \Psi} \max_{\Omega} \left\{ n^2 H^2 + 4 \int_{\Omega} u_i^2 |\rho \bar{c} - \rho f| e^{-f} dv + 2\rho \int_{\Omega} u_i^2 f e^{-f} dv + n\rho - 2\rho \bar{c} \right\},$$

and

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

By the recursion formula of Cheng and Yang, we get

$$\lambda_{k+1} + c \leq (1 + \frac{4}{n})(\lambda_1 + c) k^{2/n}.$$

By the argument in [69], we know that  $M^n$  is isometric to  $\mathbb{R}^n$ . Therefore, by the same method as in the proof of theorem 1.4, we can get

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}},$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}},$$

$C_0(n)$  is the same as the one in (1.6). This finishes the proof of this corollary.  $\square$

**Corollary 4.7.** *Let  $(M^n, g, f)$  be a complete, expanding, Ricci soliton. If the scalar curvature  $S \geq 0$  and  $S \in L^1(M^n, e^{-f} dv)$ . Assume that  $\lambda_i$  denotes the  $i$ -th eigenvalue of Dirichlet problem Dirichlet problem (1.17) of the drifting Laplacian. Then, for any  $k = 1, 2, \dots$ , one has*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \tag{4.24}$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}}, \tag{4.25}$$



$C_0(n)$  is the same as the one in (1.6), where

$$c = \frac{1}{4} \left( n\rho + 2\rho\bar{c} - 2\rho \max_{\Omega} f \right),$$

and

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

**Proof.** Since the scalar curvature  $S \geq 0$ , we have

$$\begin{aligned} & \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} \left( n^2 H^2 + 4|\rho f - \rho\bar{c}| + 2\rho f + n\rho - 2\rho\bar{c} - S \right) \\ & \leq \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} \left( n^2 H^2 + 4|\rho f - \rho\bar{c}| + 2\rho f + n\rho - 2\rho\bar{c} \right), \end{aligned}$$

where

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

By the assumption of this corollary, we know that  $M^n$  is isometric to the standard Euclidean space [69]. Since  $\rho < 0$  (i.e.,  $(M^n, g, f)$  is an expanding Ricci soliton) and eigenvalues is invariant in the sense of isometries, we have the following eigenvalue inequality:

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (4.26)$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}}, \quad (4.27)$$

$C_0(n)$  is the same as the one in (1.6), where

$$c = -\frac{\rho}{4} \inf_{\psi \in \Psi} \max_{\Omega} (4|f - \bar{c}| - 2f - n + 2\bar{c}),$$

where

$$\bar{c} = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

Thus, it completes the proof of this corollary. □

Let  $S_* = \inf_{M^n} S$ . Assume that  $(M^n, g, f)$  is a geodesically complete expanding gradient Ricci soliton, then we have

**Corollary 4.8.** *Let  $(M^n, g, f)$  be an  $n$ -dimensional, geodesically complete, expanding gradient Ricci soliton. If  $S_* \in (-\infty, n\rho) \cup (0, +\infty)$  or  $S(x) \leq n\rho$ . Assume that  $\lambda_i$  denotes the  $i$ -th eigenvalue of Dirichlet problem (1.17) of the drifting Laplacian. Then, for any  $k = 1, 2, \dots$ , one has*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}}, \quad (4.28)$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (n^2 H^2),$$

$$C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}},$$

$C_0(n)$  is the same as the one in (1.6).

**Proof.** Under the assumption of this corollary, and  $S_* \in (-\infty, n\rho) \cup (0, +\infty)$  or  $S(x) \leq n\rho$  is Einstein and the soliton is trivial. Consequently, we have

$$|\nabla f| = 0, \quad (4.29)$$

and

$$\Delta f = 0. \quad (4.30)$$

Furthermore, substituting (4.29) and (4.30) into (2.16), we obtain (4.28). This completes the proof of this corollary.  $\square$

Suppose that  $(M^n, g, f)$  is a complete shrinking Ricci soliton, then  $S(x) > 0$  on  $M^n$  unless  $S(x) \equiv 0$  on  $M^n$ , and  $M^n$  is isometric to  $\mathbb{R}^n$  (see [69]). It is well known that eigenvalues is invariant in the sense of isometries, therefore, we prove the following:

**Corollary 4.9.** *Let  $(M^n, g, f)$  be an  $n$ -dimensional, complete shrinking Ricci soliton with scalar curvature function  $S(x) \leq 0$  on  $M^n$ . Assume that  $\lambda_i$  denotes the  $i$ -th eigenvalue of Dirichlet problem (1.17) of the drifting Laplacian. Then, for any  $k = 1, 2, \dots$ , one has*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega,f} k^{\frac{1}{n}}, \quad (4.31)$$

where

$$C_{n,\Omega,f} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}},$$

$C_0(n)$  is the same as the one in (1.6), and

$$c = \frac{1}{4} \max_{\Omega} (4|\rho f - \rho \bar{c}| + 2\rho f + n\rho - 2\rho \bar{c}),$$

and

$$c = \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

Let  $(M^n, g, f)$  be complete, gradient (or expanding) Ricci soliton with  $|\nabla f| \in L^p(M^n, e^{-f} dv)$ , where  $1 \leq p \leq +\infty$ , then, we have  $\nabla f = 0$  [69]. Then, one can easily prove the following:

**Corollary 4.10.** *Let  $(M^n, g, f)$  be a complete gradient shrinking (or expanding) Ricci soliton with nonnegative Ricci curvature, and contains a line. Assume that where  $|\nabla f| \in L^p(M^n, e^{-f} dv)$ , and  $1 \leq p \leq +\infty$ . Assume that  $\lambda_i$  denotes the  $i$ -th eigenvalue of Dirichlet problem Dirichlet problem (1.17) of the drifting Laplacian. Then, for any  $k = 1, 2, \dots$ , one has*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}}, \quad (4.32)$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (n^2 H^2),$$

$$C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}},$$

$C_0(n)$  is the same as the one in (1.6).

The following theorem is to give an intrinsic eigenvalue inequality of drifting Laplacian on Ricci solitons.

**Theorem 4.11.** *Let  $(M^n, g, f)$  be a complete gradient shrinking Ricci soliton with nonnegative Ricci curvature, and contains a line. Assume that  $\lambda_i$  is the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the Dirichlet problem (1.17), then we have*

$$\lambda_{k+1} - \lambda_k \leq 4 \sqrt{C_0(1)} (\lambda_1 + c) k,$$

where  $C_0(1)$  is the case that  $n = 1$  in (1.6),  $\kappa = \frac{1}{2} \text{Diameter}(\Omega) + 2 \min_{x \in \Omega} \sqrt{f(x)} + \bar{c}$ ,

$$c = \frac{1}{4} \left( \frac{\rho}{2} \kappa^2 + 2 \sqrt{2\rho\kappa} \lambda_i^{\frac{1}{2}} \right),$$

and

$$\bar{c} = n\rho - 2\rho \frac{\int_{\Omega} f e^{-f} dv}{\int_{\Omega} e^{-f} dv}.$$

**Proof.** Assume that  $\gamma$  is a geodesic line on  $(M^n, g, f)$  and  $B^+(x)$  is Busemann function associated with  $\gamma$ . and  $h(x) = B^+(x)$ . Then, we have

$$|\nabla B^+(x)|^2 = 1$$

a.e. in  $\Omega$ . In (2.16), we suppose that  $l = 1$ ,

$$\frac{a_1^2 + b_1}{2} (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \|2\langle \nabla F_1, \nabla u_i \rangle + u_i \Delta_f F_1\|^2, \quad (4.33)$$

Putting  $F_1(x) = B^+(x)$  and substituting into the inequality (4.33), we have

$$a_1^2 = \|\nabla F_1 u_i\|_{\Omega}^2 = \sqrt{\|\nabla F_1\|^2 u_i^2}_{\Omega} = b_1 = 1,$$

which implies that

$$(\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \|u_i \Delta_f B^+ + 2\langle \nabla B^+, \nabla u_i \rangle\|_{\Omega}^2, \quad (4.34)$$

Using equation (1.21) and the contracted second Bianchi identity, we have (see [7] or Theorem 20.1 in [33])

$$R + |\nabla f|^2 - 2\rho f = \bar{c} \quad (4.35)$$

for some constant  $\bar{c}$ . Here  $R$  denotes the scalar curvature of  $g_{ij}$ . By Lemma 2.3 in [8], we know that

$$|\nabla f|^2 \leq \frac{\rho}{2} (r(x) + 2\sqrt{f(x_0)} + \bar{c})^2. \quad (4.36)$$

Here  $r(x) = d(x_0, x)$  is the distance function from some fixed point  $x_0 \in M^n$ . Since  $\Delta B^+(x) = 0$  (see [70]), it follows from (4.35) and (4.36) that (cf. [85])

$$\|u_i \Delta_f B^+ + 2\langle \nabla B^+, \nabla u_i \rangle\|_{\Omega}^2 \leq 4\lambda_i + \frac{\rho}{2} \kappa^2 + 2\sqrt{2\rho\kappa} \lambda_i^{\frac{1}{2}}, \quad (4.37)$$

where  $\kappa = \frac{1}{2} \text{Diameter}(\Omega) + 2 \min_{x \in \Omega} \sqrt{f(x)} + \bar{c}$ , and  $\bar{c}$  is a constant satisfies  $R + |\nabla f|^2 - f = \bar{c}$ . Inserting (4.37) into (4.34), we get

$$(\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \left( 4\lambda_i + \frac{\rho}{2} \kappa^2 + 2\sqrt{2\rho\kappa} \lambda_i^{\frac{1}{2}} \right). \quad (4.38)$$

Putting

$$c = \frac{1}{4} \left( \frac{\rho}{2} \kappa^2 + 2\sqrt{2\rho\kappa} \lambda_i^{\frac{1}{2}} \right),$$

then we have

$$\lambda_{k+2} - \lambda_{k+1} \leq 4 \sqrt{(\lambda_{k+2} + c)(\lambda_1 + c)}. \quad (4.39)$$

In [85], the author proved the following eigenvalue inequality:

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left( 4\lambda_i + \frac{\rho}{2} k^2 + 2 \sqrt{2\rho\kappa} \lambda_i^{\frac{1}{2}} \right). \quad (4.40)$$

Therefore, by Cheng and Yang's recursion formula, we have

$$\lambda_{k+1} + c \geq C_0(1) (\lambda_1 + c) k^2. \quad (4.41)$$

Synthesizing (4.40) and (4.41), we yield

$$\begin{aligned} \lambda_{k+2} - \lambda_{k+1} &\leq 4 \sqrt{(\lambda_{k+2} + c)(\lambda_1 + c)} \\ &\leq 4 \sqrt{C_0(1) (\lambda_1 + c) (k+1)}, \end{aligned}$$

where  $C_0(1)$  is the same as (1.6). Hence, we finish the proof of this theorem.  $\square$

## 5 Applying to Some Important Solitons

It is well known that nontrivial compact Ricci solitons may exist only when the dimension of  $M^n$  is larger than 3. and they must have nonconstant positive scalar curvature ( $S > 0$ ). However, complete noncompact examples exist in any dimension as the Gaussian soliton (i.e., the radial vector field on the Euclidean space) shows. In this section, the first object we consider is the Gaussian soliton, which is introduced by Hamilton in [32]. It is not difficult to check that the flat Euclidean space  $(\mathbb{R}^n, \delta_{ij})$  is a gradient shrinker with potential function  $f = |x|^2/4$ :

$$\partial_i \partial_j f = \frac{1}{2} \delta_{ij}.$$

The Gaussian shrinking soliton is exactly the triple  $(\mathbb{R}^n, \delta_{ij}, |x|^2/4)$ . We investigate the eigenvalue of drifting Laplacian on the Gaussian soliton and prove the following:

**Theorem 5.1.** *Let  $(M^n, g, f)$  be the Gaussian shrinking soliton and  $\lambda_i$  be the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the eigenvalue problem (1.17). Then,*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}},$$

where

$$C_{n,\Omega} = 4(\lambda_1 + c_1) \sqrt{\frac{C_0(n)}{n}}.$$

**Proof.** Let  $F_j(x) = x^j$ , where  $x^j$  denotes the  $j$ -th local coordinate of  $x_0 \in \Omega \subset \mathbb{R}^n$ . Hence, we have  $|\nabla x^j| = 1$  and  $\Delta x^j = 0$ . Let  $l = n$ , then, from (2.16), we have

$$a_j = \sqrt{\|\nabla F_j u_i\|^2} = \sqrt{\|\nabla x_j u_i\|^2} = 1,$$

$$b_j = \sqrt{\| |\nabla F_j|^2 u_i \|^2} = \sqrt{\| |\nabla x_j|^2 u_i \|^2} = 1.$$

Therefore, we have

$$a_j = b_j,$$

and

$$\sum_{j=1}^n (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \sum_{j=1}^n \|2\langle \nabla x_j, \nabla u_i \rangle + u_i \Delta_f x_j\|^2. \quad (5.1)$$

Since

$$\begin{aligned} \sum_{j=1}^n \|2\langle \nabla x_j, \nabla u_i \rangle + u_i \Delta_f x_j\|^2 &= 4 \int_{\Omega} \langle \nabla x^\alpha, \nabla u_1 \rangle e^{-f} dv - \int_{\Omega} \langle \nabla f, \nabla x^\alpha \rangle^2 u_1^2 e^{-f} dv \\ &\quad + 2 \int_{\Omega} \langle \nabla \langle \nabla f, \nabla x^\alpha \rangle, \nabla x^\alpha \rangle u_1^2 e^{-f} dv, \end{aligned} \quad (5.2)$$

and

$$\sum_{\alpha=1}^n \langle \nabla x^\alpha, \nabla x^\alpha \rangle = n,$$

from (5.2), we have

$$\begin{aligned}
n(\lambda_{k+2} - \lambda_{k+1})^2 &\leq 16(\lambda_{k+2} + c) \sum_{\alpha=1}^n \left( \int_{\Omega} \langle \nabla x^{\alpha}, \nabla u_1 \rangle e^{-f} dv - \frac{1}{4} \int_{\Omega} \langle \nabla f, \nabla x^{\alpha} \rangle^2 u_1^2 e^{-f} dv \right. \\
&\quad \left. + \frac{1}{2} \int_{\Omega} \langle \nabla \langle \nabla f, \nabla x^{\alpha} \rangle, \nabla x^{\alpha} \rangle u_1^2 e^{-f} dv \right) \\
&= 16(\lambda_{k+2} + c) \left( \lambda_1 - \frac{1}{4} \int_{\Omega} \langle \nabla \left( \frac{|x|^2}{4} \right), \nabla \left( \frac{|x|^2}{4} \right) \rangle u_1^2 e^{-f} dv \right. \\
&\quad \left. + \frac{1}{2} \sum_{\alpha=1}^n \int_{\Omega} \langle \nabla \langle \nabla \left( \frac{|x|^2}{4} \right), \nabla x^{\alpha} \rangle, \nabla x^{\alpha} \rangle u_1^2 e^{-f} dv \right) \\
&= 16(\lambda_{k+2} + c) \left( \lambda_1 - \frac{1}{16} \int_{\Omega} |x|^2 u_1^2 e^{-f} dv + \frac{n}{4} \int_{\Omega} u_1^2 e^{-f} dv \right) \\
&\leq (\lambda_{k+2} + c) \left( 16\lambda_1 + 4n - \min_{\Omega} |x|^2 \right) \\
&= 16(\lambda_{k+2} + c) (\lambda_1 + c).
\end{aligned} \tag{5.3}$$

Therefore, we deduce from (3.17) that,

$$\begin{aligned}
\lambda_{k+2} - \lambda_{k+1} &\leq 4 \sqrt{\frac{\lambda_{k+2} + c}{n}} (\lambda_1 + c) \\
&\leq 4(\lambda_1 + c) \sqrt{\frac{C_0(n)}{n}} (k+1)^{\frac{1}{n}} \\
&= C_{n,\Omega} (k+1)^{\frac{1}{n}},
\end{aligned}$$

where

$$C_{n,\Omega} = 4(\lambda_1 + c) \sqrt{\frac{C_0(n)}{n}}.$$

This completes the proof of Theorem 1.2.  $\square$

We suppose that  $(\mathbb{N}^m, \langle \cdot, \cdot \rangle)$  is any  $m$ -dimensional Einstein manifold with Ricci curvature  $Ric(\mathbf{w}) \neq 0$ ,  $\mathbf{w} \in \mathbb{N}^m$ , and  $f(\mathbf{v}, \mathbf{w}) : \mathbb{R}^{n-m} \times \mathbb{N}^m \rightarrow \mathbb{R}$  is defined by (cf. [69])

$$f(\mathbf{v}, \mathbf{w}) = \frac{Ric(\mathbf{w})}{2} |\mathbf{v}|_{\mathbb{R}^{n-m}}^2 + \langle \mathbf{v}, \mathbf{B} \rangle_{\mathbb{R}^{n-m}} + \mathbf{C}, \tag{5.4}$$

with  $\mathbf{C} \in \mathbb{R}$  and  $\mathbf{B} \in \mathbb{R}^{n-m}$ , where  $|\cdot|_{\mathbb{R}^{n-m}}$  denotes the standard inner on the  $(n-m)$ -dimensional Euclidean space  $\mathbb{R}^{n-m}$ . Then, it is well known that the Riemannian product manifold

$$(\mathbb{R}^{n-m} \times \mathbb{N}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^{n-m}} + \langle \cdot, \cdot \rangle_{\mathbb{N}^m}, f)$$

is a (noncompact) shrinking soliton. In particular, we consider the unit round cylinder  $\mathbb{S}^m(1) \times \mathbb{R}^{n-m}$  which is a noncompact shrinking soliton, and assume that  $\mathbf{B} = \mathbf{0} \in \mathbb{R}^{n-m}$  and  $\mathbf{C} = 0$ , i.e., by substituting them into (5.4), we have

$$f(\mathbf{v}, \mathbf{w}) = \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2}. \quad (5.5)$$

Under the above assumption, we can prove the following theorem.

**Theorem 5.2.** *Let*

$$(\mathbb{R}^{n-m} \times \mathbb{N}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^{n-m}} + \langle \cdot, \cdot \rangle_{\mathbb{N}^m}, f)$$

*be an  $n$ -dimensional, gradient Ricci soliton, with*

$$f(\mathbf{v}, \mathbf{w}) = \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2}.$$

*Let  $\lambda_i$  be the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the eigenvalue problem (1.17). Then, we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}},$$

*where*

$$C_{n,\Omega} = 4(\lambda_1 + c_1) \sqrt{\frac{C_0(n)}{n}}.$$

This proof is given in the appendix.

*Remark 5.1.* In the above theorem, we consider the special case:  $\mathbb{R}^{n-m} \times \mathbb{N}^m = \mathbb{R}^{n-m} \times \mathbb{S}^m$ ,  $\mathbf{B} = \mathbf{0} \in \mathbb{R}^{n-m}$  and  $\mathbf{C} = \mathbf{0}$ . However, for the general case, we can obtain similar eigenvalue inequality of Dirichlet problem (1.17) by the same argument as in the proof of theorem 5.2.

*Remark 5.2.* Suppose that the dimension  $n \geq 3$ , and  $(M^n, g, f, \cdot)$  is a complete, rotationally invariant shrinking soliton structure on a manifold  $M^n$ , which is diffeomorphic to one of  $\mathbb{S}^n$ ,  $\mathbb{R}^n$ , or  $\mathbb{R} \times \mathbb{S}^{n-1}$ . Then, one has (cf. [45])

- (1) if  $M^n \cong \mathbb{S}^n$ , then  $g$  is isometric to a round sphere and  $f \equiv \text{const}$ ;
- (2) if  $M^n \cong \mathbb{R}^n$ , then  $g$  is flat;
- (3) if  $M^n \cong \mathbb{R} \times \mathbb{S}^{n-1}$ , then  $g$  is isometric to the standard cylinder  $dr^2 + \omega_0^2 g_{\mathbb{S}^{n-1}}$  of radius  $\omega_0 = \sqrt{(n-2)/\rho}$  and  $f = f(r) = (n-2)r^2/(2\omega_0^2) + \text{linear}$ . According to the above classification of solitons, it is not difficult to obtain similar upper bound of the consecutive eigenvalues of drifting Laplacian on those complete, rotationally invariant shrinking solitons since eigenvalues are invariant in the sense of isometry.

## 6 Eigenvalues of Drifting Laplacian on Self-shrinkers

In this section, we consider that  $X : M^n \rightarrow \mathbb{R}^{n+p}$  is an  $n$ -dimensional submanifold in the  $(n+p)$ -dimensional Euclidean space  $\mathbb{R}^{n+1}$ . Let  $\{e_1, e_2, \dots, e_n\}$  be a local orthonormal basis of  $M^n$  with respect



to the induced metric, and  $\{\theta_1, \theta_2, \dots, \theta_n\}$  be their dual 1-forms. Let  $e_{n+1}, e_{n+2}, \dots, e_{n+p}$  be the local unit orthonormal normal vector fields. Furthermore, we make the following convention on the range of indices:

$$\begin{aligned} 1 \leq i, j, k, \dots \leq n; \\ n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p. \end{aligned}$$

By Cartan lemma, we have

$$h_{ij}^\alpha = h_{ji}^\alpha (\forall \alpha, \forall i, j),$$

where  $h_{ij}^\alpha$  is the components of the second fundamental form. The second fundamental form  $h$  of  $M^n$ , the mean curvature vector  $H$  and the norm square of the second fundamental form  $A$  are defined, respectively, by

$$\begin{aligned} h &= \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n h_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha, \\ H &= \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \sum_{i=1}^n h_{ii}^\alpha e_\alpha, \end{aligned}$$

and

$$|A|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$$

be the norm square of the second fundamental form. If the position vector  $X$  evolves in the direction of the mean curvature  $H$ , then it gives rise to a solution to mean curvature flow:  $X(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$  satisfying  $X(\cdot, 0) = X(\cdot)$  and

$$\frac{\partial X(p, t)}{\partial t} = H(p, t), \quad (p, t) \in M \times [0, T),$$

where  $H(p, t)$  denotes the mean curvature vector of hypersurface  $M_t = X(M^n, t)$  at point  $X(p, t)$ . In this section, we consider the self-shrinker of the mean curvature flow, which is introduced by Huisken in [41](cf. Colding and Minicozzi [23]). An  $n$ -dimensional submanifold  $M^n$  in the Euclidean space  $\mathbb{R}^{n+p}$  is called a self-shrinker if it satisfies

$$n\vec{H} = -X^N,$$

where  $\vec{H}$  and  $X^N$  denote the mean curvature vector and the orthogonal projection of  $X$  into the normal bundle of  $M^n$ , respectively.

**Theorem 6.1.** *Under the assumption of theorem 5.2, we have*

$$\lambda_{k+1} + c \leq \left(1 + \frac{4}{n}\right)(\lambda_1 + c) k^{2/n}, \quad (6.1)$$

where  $c$  is the same constant as in the theorem 5.2.

**Proof of theorem 1.5.** Since  $M^n$  is a submanifold in the Euclidean space  $\mathbb{R}^{n+p}$ , we have

$$\Delta X = n\vec{H}.$$

Hence,

$$\begin{aligned}\Delta f &= \langle X, \Delta X \rangle + n = n - n^2 H^2, \\ |\nabla f|^2 &= |X|^2 - |X^N|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}n^2 H^2 + 2|\Delta_f f| + |\nabla f|^2 &= n^2 H^2 + |2n - |X|^2| + |X|^2 - |X^N|^2 \\ &\leq n^2 H^2 + |2n - |X|^2| + |X|^2.\end{aligned}\tag{6.2}$$

By (6.2), one can yield

$$\begin{aligned}\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 &\leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left( \lambda_i + \frac{1}{4} \int_{M^n} u_i^2 (2n - |X|^2) e^{-\frac{|X|^2}{2}} dv \right) \\ &\leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left( \lambda_i + \frac{1}{4} \int_{M^n} u_i^2 (n^2 H^2 + |2n - |X|^2| + |X|^2) e^{-\frac{|X|^2}{2}} dv \right) \\ &\leq \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (\lambda_i + c),\end{aligned}\tag{6.3}$$

where

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (n^2 H^2 + |2n - |X|^2| + |X|^2),$$

and  $\Psi$  denotes the set of all isometric immersions from  $M^n$  into a Euclidean space. Here, we note that the first inequality (6.3) is established in [85]. Therefore, it follows from Cheng and Yang's recursion formula (see [19]) that,

$$\lambda_{k+1} + c \leq C_0(n) (\lambda_1 + c) k^{\frac{2}{n}},\tag{6.4}$$

Let  $F_j(x) = \alpha_j x^j$  and  $\alpha_j > 0$ , such that

$$a_j^2 = \|\nabla F_j u_i\|_{\Omega}^2 \geq \sqrt{\|\nabla F_j\|^2 \|u_i\|_{\Omega}^2} = b_j \geq 0,$$

$$\sum_{j=1}^{n+p} \int 2u_i \langle \nabla F_j, \nabla f \rangle \Delta F_j e^{-f} dv = 0,$$

and

$$\sum_{j=1}^{n+p} \int 2u_i \langle \nabla F_j, \nabla u_i \rangle \Delta F_j e^{-f} dv = 0,$$

where  $j = 1, 2, \dots, n+p$ , and  $x^j$  denotes the  $j$ -th standard coordinate function of the Euclidean space  $\mathbb{R}^{n+p}$ . Let

$$\alpha = \min_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\bar{\alpha} = \max_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\beta = \min_{1 \leq j \leq n+p} \{b_j\}.$$

According to theorem 1.2, lemma 3.1 and (6.2), we have

$$\begin{aligned} \sum_{j=1}^l \frac{a_j^2 + b_j}{2} &= \sum_{j=1}^{n+p} \frac{a_j^2 + b_j}{2} \\ &\geq \frac{1}{2} \left( n\alpha^2 + \sum_{j=1}^{n+p} b_j \right) \\ &\geq \frac{1}{2} (n\alpha^2 + (n+p)\beta), \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \sum_{j=1}^{n+p} \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{\Omega}^2 &\leq \bar{\alpha}^2 \left( 4\lambda_i + \int_{\Omega} u_i^2 (|\nabla f|^2 + 2|\Delta_f f| + n^2 H^2) e^{-f} dv \right) \\ &\leq 4\bar{\alpha}^2 (\lambda_i + c). \end{aligned} \tag{6.6}$$

Let  $i = 1, \tau = c$ , then, by proposition 2.4, we have

$$(\lambda_{k+2} - \lambda_{k+1})^2 \leq \frac{32\bar{\alpha}^2(\lambda_{k+2} + c)}{n\alpha^2 + (n+p)\beta} (\lambda_1 + c), \tag{6.7}$$

Therefore, we deduce from (6.7) that,

$$\begin{aligned} \lambda_{k+2} - \lambda_{k+1} &\leq \sqrt{\frac{32\bar{\alpha}^2}{n\alpha^2 + (n+p)\beta}} \sqrt{\lambda_1 + c} \sqrt{\lambda_{k+2} + c} \\ &\leq (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}} (k+1)^{\frac{1}{n}} \\ &= C_{n,\Omega} (k+1)^{\frac{1}{n}}, \end{aligned}$$

where

$$C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}}.$$

Thus, we complete the proof of this theorem.  $\square$

**Remark 6.1.** Let  $M^n$  be an  $n$ -dimensional complete minimal self-shrinker in the  $(n+p)$ -dimensional Euclidean space  $\mathbb{R}^{n+p}$ , eigenvalues of the Dirichlet problem (1.17) of drifting Laplacian with  $f = \frac{|X|^2}{2}$  then the constant  $c$  in theorem 7.8 will be written as

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (|2n - |X|^2| + |X|^2).$$

**Theorem 6.2.** Let  $(M^n, g)$  be an  $n$ -dimensional, compact Riemannian manifold. Let  $H$  and  $X$  denote the mean curvature of  $M^n$  and the position vector of  $M^n$ , respectively, and  $\lambda_i$  be the  $i$ -th ( $i = 0, 1, 2, \dots, k$ ) eigenvalue of the eigenvalue problem (1.19). Then, we have

$$\bar{\lambda}_{k+1} - \bar{\lambda}_k \leq C_{n,\Omega} (k+1)^{\frac{1}{n}},$$

where

$$C_{n,\Omega} = (\bar{\lambda}_1 + c) \sqrt{\frac{32\bar{\alpha}^2 C_0(n)}{n\alpha^2 + (n+p)\beta}},$$

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} (n^2 H^2 + |2n - |X|^2| + |X|^2),$$

and  $\Psi$  denotes the set of all isometric immersions from  $M^n$  into a Euclidean space.

**Remark 6.2.** Let  $M^n$  be an  $n$ -dimensional complete self-shrinker with polynomial volume growth in  $\mathbb{R}^{n+1}$ . In [17], Q.-M. Cheng and G. Wei proved the fact: If  $M^n$  is noncompact manifolds with  $A \leq 10/7$ , which can split into at most  $m$  geodesic lines, then it is isometric to the hyperplane  $\mathbb{R}^n$  when  $m = n$ , and  $M^n$  is isometric to a cylinder  $\mathbb{R}^m \times \mathbb{S}^{n-m}(\sqrt{n-m})$ , for  $1 \leq m \leq n-1$ ; If  $M^n$  is compact, then it is isometric to the round sphere  $\mathbb{S}^n(\sqrt{n})$ . Therefore, according to theorem 6.2 and the result of classification of self-shrinkers, it is not difficult to estimate the upper bound for the gap of the consecutive eigenvalues Dirichlet problem (1.17) of Laplacian with  $f = \frac{|X|^2}{2}$ .

**Theorem 6.3.** Let  $(M^n, g)$  be an  $n$ -dimensional, compact Riemannian manifold and  $\lambda_i$  be the  $i$ -th ( $i = 0, 1, 2, \dots, k$ ) eigenvalue of the eigenvalue problem (1.8) with  $f = \frac{|X|^2}{4}$ . Then, for any  $h \in C^3(\Omega) \cap C^2(\partial\Omega)$ , we have

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{\frac{1}{n}},$$

where

$$C_{n,\Omega} = 4(\lambda_1 + c_1) \sqrt{\frac{C_0(n)}{n}}.$$

*Remark 6.3.* Let  $M^2$  be a 2-dimensional complete self-shrinker in  $\mathbb{R}^3$  with constant squared norm of the second fundamental form, Cheng and Ogata gave a complete classification as follows (see [15]):  $M^2$  is isometric to  $\mathbb{R}^2$ , or a cylinder  $\mathbb{S}^1(1) \times \mathbb{R}$ , or the round sphere  $\mathbb{S}^2(2)$ . Given some topological conditions (for example, the manifold can be made such assumption that it is compact or can be splitted into one geodesic line and so on), by theorem 6.2, one can obtain the similar estimates for the gap of consecutive eigenvalues of Dirichlet problem (1.17) of drifting Laplacian with  $f = \frac{|x|^2}{4}$  since eigenvalues is isometrically invariant.

## 7 Eigenvalues on the Complete Product Riemannian Manifolds

We let  $M^n$  be an  $n$ -dimensional complete Riemannian manifold with  $\infty$ -Bakry-Emery curvature  $\text{Ric}^f \geq 0$ , and  $f \in C^2(M^n)$  be bounded above uniformly on  $M^n$ . Under those assumption, F.-Q. Fang, X.-D. Li and Z.-L. Zhang [25] proved that it splits isometrically as  $\mathbb{N}^{n-m} \times \mathbb{R}^m$ , where  $\mathbb{N}^{n-m}$  is some complete Riemannian manifold without lines and  $\mathbb{R}^m$  is the  $m$ -dimensional Euclidean space. Therefore, based on the above argument, we can prove the following:

**Proposition 7.1.** *Let  $(M^n, g, d\mu)$  be an  $n$ -dimensional complete metric measure space with  $\infty$ -Bakry-Emery curvature  $\text{Ric}^f \geq 0$  and  $f \in C^2(M^n)$  be bounded above uniformly on  $M^n$ . Assume that  $\lambda_i$  is the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the Dirichlet problem (1.17), then there exists a positive integer  $m$ , where  $1 \leq m \leq n$ , such that*

$$\lambda_{k+1} - \lambda_k \leq C(m, \Omega, k) k^{\frac{1}{m}}, \quad (7.1)$$

where

$$C(m, \Omega, k) = \sqrt{\frac{4(C_0(m)(\lambda_1 + c_2))}{m}} \cdot \sqrt{4\lambda_1 + \max_{\Omega}\{|\nabla f|^2\} + 4m \max_{\Omega}\{|\nabla f|\} \sqrt{C_0(m)(\lambda_1 + c_2)} k^{\frac{1}{m}}},$$

and

$$c_2 = \frac{1}{4} \max_{\Omega}\{|\nabla f|^2\} + m \max_{\Omega}\{|\nabla f|\} \lambda_{k+1}^{\frac{1}{2}}.$$

**Proof.** Since the  $\infty$ -Bakry-Emery curvature  $\text{Ric}^f$  is nonnegative and  $f \in C^2(M^n)$  is bounded above uniformly on  $M^n$ , by Theorem 1.1 in [25], we know that the Bakry-Émery-Hadamard manifold splits isometrically as  $\mathbb{N}^{n-m} \times \mathbb{R}^m$ , where  $\mathbb{N}^{n-m}$  is some complete Riemannian manifold without lines and  $\mathbb{R}^m$  is the  $m$ -dimensional Euclidean space. Since the eigenvalue of the Dirichlet problem is an invariant of isometries, the remainder part of the proof is only to show that the inequality (7.1) holds on Bakry-Émery-Hadamard product manifolds  $\mathbb{R}^m \times \mathbb{N}^{n-m}$ . For any  $j = 1, 2, \dots, m$ , let

$$F_j = h_j(\mathbf{x}, \mathbf{y}) = h_j((x^1, x^2, \dots, x^m), y) = x^j,$$

where  $\mathbf{x} = (x^1, x^2, \dots, x^m)$  and  $x^j$  is the  $j$ -th coordinate function. Then, we have

$$\Delta_f h_j(\mathbf{x}, \mathbf{y}) = \Delta x^j + \langle \nabla f, \nabla x^j \rangle = \langle \nabla f, \nabla x^j \rangle \leq |\nabla f|, \quad (7.2)$$

$$|\nabla h_j(\mathbf{x}, \mathbf{y})|^2 = 1, \quad (7.3)$$

$$\sum_{p=1}^m \langle \nabla h_j(\mathbf{x}, \mathbf{y}), \nabla u_i \rangle^2 \leq \sum_{j=1}^m \langle \nabla u_i, \nabla u_i \rangle. \quad (7.4)$$

Hence, we have  $|\nabla x^j| = 1$  and  $\Delta x^j = 0$ . Let  $l = n$ , then, from (2.16), we have

$$a_j = \sqrt{\|\nabla F_j u_i\|_\Omega^2} = \sqrt{\|\nabla x_j u_i\|_\Omega^2} = 1,$$

$$b_j = \sqrt{\| |\nabla F_j|^2 u_i \|_\Omega^2} = \sqrt{\| |\nabla x_j|^2 u_i \|_\Omega^2} = 1.$$

Thus, we have

$$a_j = b_j.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{j=1}^m \|u_i \Delta_f h_j(\mathbf{x}, \mathbf{y}) + 2\langle \nabla h_j(\mathbf{x}, \mathbf{y}), \nabla u_i \rangle\|_\Omega^2 \leq 4\lambda_i + \max_\Omega\{|\nabla f|^2\} + 4m \max_\Omega\{|\nabla f|\}\lambda_i^{\frac{1}{2}}. \quad (7.5)$$

Let  $l = m$ ,  $F_j = h_j(\mathbf{x}, \mathbf{y})$ , by proposition 2.4, we have

$$\sum_{j=1}^m \frac{a_j^2 + b_j}{2} (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \tau) \sum_{j=1}^m \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|^2,$$

which implies that

$$m(\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \tau) \sum_{j=1}^m \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|^2. \quad (7.6)$$

From (7.5) and (7.6), it is not difficult to see that, for any  $i = 1, 2, \dots$ ,

$$\lambda_{k+2} - \lambda_{k+1} \leq \sqrt{\frac{4(\lambda_{k+2} + \tau)}{m}} \cdot \sqrt{4\lambda_i + \max_\Omega\{|\nabla f|^2\} + 4m \max_\Omega\{|\nabla f|\}\lambda_i^{\frac{1}{2}}}. \quad (7.7)$$

Recall that the author proved the following eigenvalue inequality in [85]:

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{m} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (\lambda_i + c_1),$$

where

$$c_1 = \frac{1}{4} \max_\Omega\{|\nabla f|^2\} + m \max_\Omega\{|\nabla f|\}\lambda_i^{\frac{1}{2}}.$$

Therefore, we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{m} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (4\lambda_i + c_2), \quad (7.8)$$

where

$$c_2 = \frac{1}{4} \max_{\Omega} \{|\nabla f|^2\} + m \max_{\Omega} \{|\nabla f|\} \lambda_{k+1}^{\frac{1}{2}}.$$

Therefore, by Q.-M. Cheng and H.-C. Yang's recursion formula and (7.8), we yield

$$\lambda_{k+1} + c_2 \leq C_0(m) (\lambda_1 + c_2) k^{\frac{2}{m}}. \quad (7.9)$$

Putting  $\tau = c_2$ , and utilizing (7.7) and (7.9), then one can infer that

$$\begin{aligned} \lambda_{k+2} - \lambda_{k+1} &\leq \sqrt{\frac{4(\lambda_{k+2} + \tau)}{m}} \cdot \sqrt{4\lambda_1 + \max_{\Omega} \{|\nabla f|^2\} + 4m \max_{\Omega} \{|\nabla f|\} (\lambda_{k+2} + \tau)^{\frac{1}{2}}} \\ &\leq \sqrt{\frac{4(C_0(m)(\lambda_1 + c_2)(k+1)^{\frac{2}{m}})}{m}} \\ &\quad \cdot \sqrt{4\lambda_1 + \max_{\Omega} \{|\nabla f|^2\} + 4m \max_{\Omega} \{|\nabla f|\} (C_0(m)(\lambda_1 + c_2)(k+1)^{\frac{2}{m}})^{\frac{1}{2}}} \\ &= \sqrt{\frac{4(C_0(m)(\lambda_1 + c_2))}{m}} \\ &\quad \cdot \sqrt{4\lambda_1 + \max_{\Omega} \{|\nabla f|^2\} + 4m \max_{\Omega} \{|\nabla f|\} \sqrt{C_0(m)(\lambda_1 + c_2)(k+1)^{\frac{1}{m}}} \cdot (k+1)^{\frac{1}{m}}} \\ &= C(n, \Omega, k)(k+1)^{\frac{1}{m}}, \end{aligned} \quad (7.10)$$

where

$$C(m, \Omega, k) = \sqrt{\frac{4(C_0(m)(\lambda_1 + c_2))}{m}} \cdot \sqrt{4\lambda_1 + \max_{\Omega} \{|\nabla f|^2\} + 4m \max_{\Omega} \{|\nabla f|\} \sqrt{C_0(m)(\lambda_1 + c_2)(k+1)^{\frac{1}{m}}}}.$$

Hence, we complete the proof of the proposition.  $\square$

If we replace the condition of  $\text{Ric}^f \geq 0$  by  $\text{Ric}_{l,n}^f \geq 0$ , then the condition that  $f$  bounded above uniformly on  $M^n$  can be removed. Similarly, by using the same method as proposition 7.1 and noticing lemma 2.6 in [25], it is not difficult to give the proof of the following proposition:

**Proposition 7.2.** *Let  $(M^n, g, d\mu)$  be an  $n$ -dimensional, connected, complete Bakry-Émery manifold with  $l$ -Bakry-Émery curvature  $\text{Ric}_{l,n}^f \geq 0$ . Assume that  $\lambda_i$  is the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the Dirichlet problem (1.17), then there exists a positive integer  $m$ , where  $1 \leq m \leq n$ , such that*

$$\lambda_{k+1} - \lambda_k \leq C(m, \Omega, k) k^{\frac{1}{m}},$$

where

$$C(m, \Omega, k) = \sqrt{\frac{4(C_0(m)(\lambda_1 + c_2))}{m}} \cdot \sqrt{4\lambda_1 + \max_{\Omega}\{|\nabla f|^2\} + 4m \max_{\Omega}\{|\nabla f|\} \sqrt{C_0(m)(\lambda_1 + c_2)} k^{\frac{1}{m}}},$$

and

$$c_2 = \frac{1}{4} \max_{\Omega}\{|\nabla f|^2\} + m \max_{\Omega}\{|\nabla f|\} \lambda_{k+1}^{\frac{1}{2}}.$$

*Remark 7.1.* In the proofs of proposition 7.1 and proposition 7.2, the weighted coefficients are assumed that  $a_j = 1$  for any  $j = 1, 2, \dots, l$ .

*Remark 7.2.* Under the assumptions of proposition 7.1 and proposition 7.2, Riemannian manifold  $M^n$  splits isometrically as  $\mathbb{N}^{n-m} \times \mathbb{R}^m$ . Therefore, the integer  $m$  in proposition 7.1 and proposition 7.2 is exactly the dimension of the Euclidean space  $\mathbb{R}^m$ .

*Remark 7.3.* Suppose that  $(M^n, g, f)$  ( $n \geq 4$ ), is a complete, shrinking, gradient Ricci soliton with harmonic Weyl tensor, then,  $M^n = N^k \times \mathbb{R}^{n-k}$ , where  $N^k$  is an Einstein manifold (cf. [26, 62]). Therefore, by the same method as the proof of proposition 7.1, it is not difficult to obtain a similar estimate for the consecutive eigenvalues of drifting Laplacian on the soliton.

In order to generalize the trivial Ricci solitons, Petersen and Wylie introduced the notion of rigidity of gradient Ricci solitons in [67]. A gradient soliton is said to be rigid if it is isometric to a quotient of  $\mathbb{N} \times \mathbb{R}^k$  where  $\mathbb{N}$  is an Einstein manifold and  $f = \frac{\rho}{2}|x|^2$  on the Euclidean factor. That is, the Riemannian manifold  $(M^n, g)$  is isometric to  $\mathbb{N} \times_{\Gamma} \mathbb{R}^k$ , where  $\Gamma$  acts freely on  $N$  and by orthogonal transformations on  $\mathbb{R}^k$ . Rigidity of gradient Ricci solitons has been studied in [67, 68].

*Remark 7.4.* It is well known that Einstein manifolds have harmonic Weyl tensor. In fact, under some geometric implications, a Ricci soliton has the assumption of the harmonicity of the Weyl tensor. For example, F.-L. Manuel and G.-R. Eduardo [56] showed that a compact Ricci soliton is rigid if and only if it has harmonic Weyl tensor, which gives a positive answer to Problem C.2 posed in [24]. For the complete noncompact case, F.-L. Manuel and G.-R. Eduardo proved that a gradient shrinking Ricci soliton is rigid if and only if it has harmonic Weyl tensor, under the assumptions that the Ricci curvature is bounded from below and the Riemannian curvature has at most exponential growth in [56]. Therefore, by remark 7.3 and the proof of proposition 7.1, one can obtain a similar estimate for the consecutive eigenvalues of drifting Laplacian on those solitons with the above rigid and geometric conditions. Let  $(M^n, g, f)$  be an  $n$ -dimensional compact Ricci soliton with constant sectional curvature. Then, the Weyl tensor vanishes [56]. Therefore, for all of the compact Ricci soliton with constant sectional curvature, one can also obtain the similar eigenvalue inequality by the same argument. In addition, by the other classifications of Ricci solitons, for example in [62, 63], one can obtain the corresponding eigenvalue inequality of drifting Laplacian on some complete metric measure spaces.

If we consider the case that  $f$  is a constant, the drifting Laplacian is exactly the standard Laplacian on complete Riemannian manifolds. Then, one can prove the following:



**Corollary 7.3.** *Let  $(M^n, g, d\mu)$  be an  $n$ -dimensional complete Riemannian manifold with Ricci curvature  $\text{Ric} \geq 0$  and  $f \in C^2(M^n)$  be bounded above uniformly on  $M^n$ . Assume that  $\lambda_i$  is the  $i$ -th ( $i = 1, 2, \dots, k$ ) eigenvalue of the Dirichlet problem (1.1), then there exists a positive integer  $m$ , where  $1 \leq m \leq n$ , such that*

$$\lambda_{k+1} - \lambda_k \leq C(m, \Omega, k)k^{\frac{1}{m}},$$

where

$$C(m, \Omega, k) = 4\lambda_1 \sqrt{\frac{C_0(m)}{m}}.$$

## 8 Appendix

In this appendix, we give the proof of theorem 5.2.

**Proof of theorem 5.2.** We denote the position vector of the  $n$ -dimensional unit round cylinder  $\mathbb{R}^{n-m} \times \mathbb{S}^m(1)$  in  $n+1$ -dimensional Euclidean space  $\mathbb{R}^{n+1}$  by

$$\mathbf{x} = (\mathbf{v}, \mathbf{w}) = (x^1, x^2, \dots, x^{n-m}, x^{n-m+1}, x^{n-m+2}, \dots, x^n, x^{n+1}),$$

where  $\mathbf{v} = (x^1, x^2, \dots, x^{n-m})$ ,  $\mathbf{w} = (x^{n-m+1}, x^{n-m+2}, \dots, x^n, x^{n+1})$ , and then, we obtain

$$\sum_{j=n-m+1}^{n+1} (x^j)^2 = 1, \quad \sum_{j=1}^{n+1} |\nabla x^j|^2 = n, \quad (8.1)$$

and

$$\Delta x^j = \begin{cases} 0, & \text{if } j = 1, \dots, n-m, \\ -mx^j, & \text{if } j = n-m+1, \dots, n+1. \end{cases} \quad (8.2)$$

For any  $j$  ( $j = 1, 2, \dots, n+1$ ), let  $l = n+1$  and  $F_j(x) = \delta_j x^j$  and  $\delta_j > 0$ , such that

$$\sum_{j=1}^{n+1} \int_{\Omega} u_i^2 \Delta(\delta_j x^j) \langle \nabla \left( \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2} \right), \nabla(\delta_j x^j) \rangle d\mu = 0, \quad (8.3)$$

$$\begin{aligned} & (m-1) \sum_{j=1}^{n-m} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle u_i (\delta_j x^j) d\mu + m \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle u_i (\delta_j x^j) d\mu \\ &= \widetilde{\delta}^2 \left[ (m-1) \sum_{j=1}^{n-m} \int_{\Omega} \langle \nabla x^j, \nabla u_i \rangle u_i x^j d\mu - 4m \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla x^j, \nabla u_i \rangle u_i x^j d\mu \right], \end{aligned} \quad (8.4)$$

and

$$a_j^2 = \|\nabla F_j u_i\|^2 \geq \sqrt{\|\nabla F_j\|^2 u_i^2} = b_j \geq 0.$$

Let

$$\begin{aligned}\delta &= \min_{1 \leq j \leq n+p} \{\delta_j\}, \\ \bar{\delta} &= \max_{1 \leq j \leq n+p} \{\delta_j\}, \\ \gamma &= \min_{1 \leq j \leq n+p} \min_{\Omega} \{b_j\}.\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{j=1}^l \frac{a_j^2 + b_j}{2} &= \sum_{j=1}^{n+1} \frac{\sqrt{\|\nabla(\delta_j x^j) u_i\|^2} + \sqrt{\|\nabla(\delta_j x^j)|^2 u_i\|^2}}{2} \\ &\geq \frac{1}{2} \left( n\delta^2 + \sum_{j=1}^{n+1} b_j \right) \\ &\geq \frac{1}{2} (n\delta^2 + (n+1)\gamma).\end{aligned}\tag{8.5}$$

For any fixed point  $x_0 \in \Omega$ , we can find a coordinate system  $(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^{n+1})$  of the  $n$ -dimensional unit round cylinder  $\mathbb{R}^{n-m} \times \mathbb{S}^m(1)$  such that at the point  $x_0$

$$\begin{aligned}\tilde{x}^1 &= \dots = \tilde{x}^n = 0, \quad \tilde{x}^{n+1} = 1, \\ \nabla \tilde{x}^{n+1} &= 0; \\ \nabla_p x^q &= \delta_p^q \quad (p, q = 1, 2, \dots, n+1).\end{aligned}\tag{8.6}$$

In fact, we can choose a constant  $(n+1) \times (n+1)$  type orthonormal matrix  $(a_j^i)_{(n+1) \times (n+1)}$  satisfying

$$\sum_{\alpha=1}^{n+1} a_p^\alpha a_q^\alpha = \delta_{pq},$$

such that

$$x^p = \sum_{\alpha=1}^{n+1} a_\alpha^p \tilde{x}^\alpha,$$

and (8.6) is satisfied at the point  $x_0$ . Thus, at the point  $x_0$ , we have

$$\begin{aligned}\sum_{p=1}^{n+1} \langle \nabla x^p, \nabla u_i \rangle^2 &= \sum_{p,q,\alpha=1}^{n+1} a_p^\alpha a_q^\alpha \langle \nabla \tilde{x}^p, \nabla u_i \rangle \langle \nabla \tilde{x}^q, \nabla u_i \rangle \\ &= \sum_{p=1}^{n+1} \langle \nabla \tilde{x}^p, \nabla u_i \rangle^2 \\ &= \sum_{p=1}^{n+1} \langle \nabla_p u_i, \nabla_p u_i \rangle \\ &= |\nabla u_i|^2.\end{aligned}$$

Since  $x_0$  is an arbitrary point, we know that for any point  $x \in \Omega$ ,

$$\sum_{p=1}^{n+1} \langle \nabla x^p, \nabla u_i \rangle^2 = |\nabla u_i|^2.$$

On the other hand, by using (8.1), we have

$$\sum_{p=n-m+1}^{n+1} \nabla(x^p)^2 = 0, \quad (8.7)$$

and

$$\sum_{p=n-m+1}^{n+1} |\nabla x^p|^2 = - \sum_{p=1}^{n+1} x^p \Delta x^p = m. \quad (8.8)$$

Let

$$\mathfrak{U} = \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|^2 = \sum_{j=1}^{n+1} \|2\langle \nabla(\delta_j x^j), \nabla u_i \rangle + u_i \Delta_f(\delta_j x^j)\|^2. \quad (8.9)$$

Then, using (8.7) and (8.8), we deduce

$$\begin{aligned} \mathfrak{U} &= \sum_{j=1}^{n+1} \|2\langle \nabla(\delta_j x^j), \nabla u_i \rangle + u_i \Delta(\delta_j x^j) - u_i \nabla \left( \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2} \right), \nabla(\delta_j x^j)\|_{\Omega}^2 \\ &= 4 \sum_{j=1}^{n+1} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle^2 d\mu + m^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} u_i^2 (\delta_j x^j)^2 d\mu \\ &\quad + (m-1)^2 \sum_{j=1}^{n-m} \int_{\Omega} u_i^2 (\delta_j x^j)^2 d\mu - 2 \sum_{j=1}^{n+1} \int_{\Omega} u_i^2 \Delta(\delta_j x^j) \langle \nabla \left( \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2} \right), \nabla(\delta_j x^j) \rangle d\mu \\ &\quad - 4(m-1) \sum_{j=1}^{n-m} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle u_i (\delta_j x^j) d\mu - 4m \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle u_i (\delta_j x^j) d\mu. \end{aligned} \quad (8.10)$$

Furthermore, by the definitions of  $\bar{\delta}$  and  $\widetilde{\delta}$ , we have

$$\begin{aligned}
\mathfrak{A} &\leq 4\bar{\delta}^2 \sum_{j=1}^{n+1} \int_{\Omega} \langle \nabla x^j, \nabla u_i \rangle^2 d\mu + m^2 \bar{\delta}^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} u_i^2(x^j)^2 d\mu \\
&+ (m-1)^2 \bar{\delta}^2 \sum_{j=1}^{n-m} \int_{\Omega} u_i^2(x^j)^2 d\mu - 2 \sum_{j=1}^{n+1} \int_{\Omega} u_i^2 \Delta(\delta_j x^j) \langle \nabla \left( \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2} \right), \nabla(\delta_j x^j) \rangle d\mu \\
&- 4(m-1) \sum_{j=1}^{n-m} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle u_i(\delta_j x^j) d\mu - 4m \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla(\delta_j x^j), \nabla u_i \rangle u_i(\delta_j x^j) d\mu \\
&= 4\bar{\delta}^2 \lambda_i + m^2 \bar{\delta}^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} u_i^2(x^j)^2 d\mu + (m-1)^2 \bar{\delta}^2 \sum_{j=1}^{n-m} \int_{\Omega} u_i^2(x^j)^2 d\mu \\
&- (m-1) \bar{\delta}^2 \sum_{j=1}^{n-m} \int_{\Omega} \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle d\mu - m \bar{\delta}^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle d\mu \\
&= 4\bar{\delta}^2 \lambda_i + (m-1)^2 \bar{\delta}^2 \sum_{j=1}^{n+1} \int_{\Omega} u_i^2(x^j)^2 d\mu + (2m-1) \bar{\delta}^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} u_i^2(x^j)^2 d\mu \\
&- (m-1) \bar{\delta}^2 \sum_{j=1}^{n+1} \int_{\Omega} \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle d\mu - \bar{\delta}^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle d\mu \\
&= 4\lambda_i + \mathfrak{B} + (2m-1) \bar{\delta}^2,
\end{aligned} \tag{8.11}$$

where

$$\begin{aligned}
\mathfrak{B} &= (m-1)^2 \bar{\delta}^2 \sum_{j=1}^{n+1} \int_{\Omega} u_i^2(x^j)^2 d\mu - (m-1) \bar{\delta}^2 \sum_{j=1}^{n+1} \int_{\Omega} \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle d\mu \\
&- \bar{\delta}^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle d\mu \\
&= (m-1) \sum_{j=1}^{n+1} \int_{\Omega} \left( (m-1) \bar{\delta}^2 u_i^2(x^j)^2 - \bar{\delta}^2 \langle \nabla(x^j)^2, \nabla(u_i)^2 \rangle \right) d\mu \\
&= (m-1) \int_{\Omega} u_i^2 \left( (m-1) \bar{\delta}^2 |\mathbf{x}|^2 + \bar{\delta}^2 \Delta_f |\mathbf{x}|^2 \right) d\mu.
\end{aligned} \tag{8.12}$$

Uniting (8.1), (8.2), (8.7) and (8.8), we have

$$\sum_{p=1}^{n+1} \Delta(x^p)^2 = 2(n-m).$$

By a direct computation, we yield

$$\Delta_f |\mathbf{x}|^2 = 2(n-1) - 2(m-1)|\mathbf{x}|^2. \quad (8.13)$$

Substituting (8.11), (8.12) and (8.13) into (8.9), we obtain

$$\begin{aligned} & \sum_{p=1}^{n+1} \|2\langle \nabla x^p, \nabla u_i \rangle + u_i \Delta x^p - u_i \langle \nabla \left( \frac{(m-1)|\mathbf{v}|_{\mathbb{R}^{n-m}}^2}{2} \right), \nabla x^p \rangle\|_{\Omega}^2 \\ &= (m-1) \int_{\Omega} u_i^2 \left( (m-1)\bar{\delta}^2 |\mathbf{x}|^2 + \bar{\delta}^2 [2(n-1) - 2(m-1)|\mathbf{x}|^2] \right) d\mu + (2m-1)\bar{\delta}^2 + 4\bar{\delta}^2 \lambda_i \\ &= (m-1)^2 \left( \bar{\delta}^2 - 2\bar{\delta}^2 \right) \int_{\Omega} u_i^2 |\mathbf{x}|^2 d\mu + (m-1)(2n-1)\bar{\delta}^2 + (2m-1)\bar{\delta}^2 + 4\bar{\delta}^2 \lambda_i \\ &\leq (m-1)^2 \max_{\Omega} \left( \bar{\delta}^2 - 2\bar{\delta}^2 \right) |\mathbf{x}|^2 + (m-1)(2n-1)\bar{\delta}^2 + (2m-1)\bar{\delta}^2 + 4\bar{\delta}^2 \lambda_i \\ &\leq (m-1)^2 \left( \bar{\delta}^2 + 2\bar{\delta}^2 \right) + (m-1)(2n-1)\bar{\delta}^2 + (2m-1)\bar{\delta}^2 + 4\bar{\delta}^2 \lambda_i. \end{aligned} \quad (8.14)$$

On the other hand, we have

$$\cdot \quad (8.15)$$

By the recursion formula given by Q.-M. Cheng and H.-C. Yang in [?], we have Let

$$\begin{aligned} c &= \frac{1}{4\bar{\delta}^2} \left[ (m-1)^2 \left( \bar{\delta}^2 + 2\bar{\delta}^2 \right) + (m-1)(2n-1)\bar{\delta}^2 + (2m-1)\bar{\delta}^2 \right] \\ &= \frac{(m-1)^2 \left( \bar{\delta}^2 + 2\bar{\delta}^2 \right)}{4\bar{\delta}^2} + \frac{(m-1)(2n-1)\bar{\delta}^2}{4\bar{\delta}^2} + \frac{(2m-1)}{4}. \end{aligned}$$

Then, we deduce from (8.5) and (2.16) that,

$$\frac{1}{2} \left( n\delta^2 + (n+1)\gamma \right) (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \sum_{j=1}^l \|2\langle \nabla F_j, \nabla u_i \rangle + u_i \Delta_f F_j\|_{\Omega}^2. \quad (8.16)$$

Let  $\tau = c, l = n+1$ . Then, by utilizing (8.14) and (8.16), we yield

$$(\lambda_{k+2} - \lambda_{k+1})^2 \leq \frac{32\bar{\delta}^2}{(n\delta^2 + (n+1)\gamma)} (\lambda_{k+2} + c)(\lambda_1 + c). \quad (8.17)$$

Therefore, we yield

$$\begin{aligned}
\lambda_{k+2} - \lambda_{k+1} &\leq \sqrt{\frac{32\bar{\delta}^2}{(n\delta^2 + (n+1)\gamma)}} \sqrt{\lambda_{k+2} + c} \sqrt{\lambda_1 + c} \\
&\leq (\lambda_1 + c) \sqrt{\frac{32C_0(n)\bar{\delta}^2}{(n\delta^2 + (n+1)\gamma)}} (k+1)^{\frac{1}{n}} \\
&= C_{n,\Omega} (k+1)^{\frac{1}{n}},
\end{aligned}$$

where

$$C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32C_0(n)\bar{\delta}^2}{(n\delta^2 + (n+1)\gamma)}}.$$

This completes the proof of this theorem. □

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